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BIPARTITE KINGS

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Abstract

Given a quadruple $(a, m; b, n), a \geq m, b \geq n$ of non-negative integers, we show that, except for a few cases, there exists a bipartite tournament $T(A, B)$ such that $|A| = a$, $|B| = b$, A contains precisely m and B contains precisely n kings.

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A *king* of a bipartite tournament T is a vertex v such that from v to any other vertex of T there is a dipath of length ≤ 4. Many results and questiones on kings of ordinary tournaments can be found in [1], [2], [3]. The proof of existence of the king in a k-partite tournament is given in [4]. The terminology and notation is that of [4].

The following lemma, actually proved in [4], will be of use.

Lemma. *If x is a king of a bipartite tournament $T(A, B)$, $x \in A$, then each vertex y, $y \in B$ is connected from x by a dipath of length ≤ 2 unless $O(x) = O(y)$.* □

Notice that y in the last case is also the king of $T(A, B)$.

Theorem 1. *Given a bipartite tournament $T(A, B)$ whose all vertices have non-zero indegree, then each partition set of $T(A, B)$ contains at least 2 kings.*

Proof. First notice that $|A| \geq 2$ and $|B| \geq 2$. Let x (resp. y) be a vertex of A (resp. B) with the maximal outdegree. By [4] x and y are kings in T. Assume that x is the only king in A and that $x \to y$. From [4] and the lemma above we obtain:

(1) $d^+(x) > d^+(z)$ for each $z \in A$.

(2) each vertex $t \in A$ is connected from x by a 2-dipath.

(3) each vertex $s \in B$ is connected from x by 1 or 3-dipath.

According to (3) there is a 3-dipath from x to y, say $x \to u \to v \to y$ where $u \in B$, $v \in A$. We claim that v is the king too.

Let w be an arbitrary vertex of T. Three cases have to be considered.

Case 1. $w \in A$. Then $v \to y \to x \to s \to w$ is a 4-dipath from v to w, where $x \to s \to w$ is a 2-dipath existing by (2). In particular for $w = x$ we have $v \to y \to x$.

Case 2. $w \in B$. As $w \in O(v)$ is trivial we assume $w \in I(v)$. Then, by the lemma, there is a dipath $y \to t \to w$ unless $O(v) = O(y)$. In the first case we obtain the 3-dipath $v \to y \to t \to w$. In the second $I(w) = I(y)$ holds implying $v \to w$, a contradiction to $w \in I(v)$.

So, v is the king in T. Applying similar arguments one can show that u is the king also. This completes the proof. □

A bipartite tournament $T(A, B)$ is said to be of the *type* $(a, m; b, n)$ if $|A| = a$, $|B| = b$, A contains precisely m and B contains precisely n kings.

Theorem 2. *Given non-negative integers a, m, b, n ($a \geq b > 0$, $a \geq m$, $b \geq n$), then there exist a bipartite tournament of the type $(a, m; b, n)$ except for the cases:*

(a) $(a, 1; b, 1)$

(b) $(1, 1; b, n)$, $n > 0$

(c) $(a, m; 1, 1)$, $m > 0$

(d) $(1, 0; 1, 0)$.

Proof. Exceptional cases (b), (c), (d) are obvious while case (a) is an immediate consequence of Theorem 1. In the following we construct examples of corresponding bipartite tournaments for each characteristic type. All non-specific arcs in these examples are assumed to be oriented from B to A.

1° $(a, 0; b, 0)$. Assume that $a \geq 2$. Then $T(A, B)$, with $A \to B$ is a required tournament.

2° $(a, 1; b, 0)$. Let $A = \{v_1, \ldots, v_a\}$ and $B = \{w_1, \ldots, w_b\}$. We define the bipartite tournament $T(A, B)$ as follows

$$v_1 \to B.$$

It is easy to see that v_1 is the only king in $T(A, B)$.

3° $(a, 0; b, 1)$. Similar to 2°.

4° $(a, m; b, n)$, $m \geq 2$, $n \geq 2$. Let $A = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_a\}$ and $B = \{w_1, \ldots, w_n, v_{n+1}, \ldots, w_b\}$. Without loss of generality we may assume tha $m \geq n$. Let $T(A, B)$ be the bipartite tournament defined as

$$\begin{array}{rcl} v_i & \to & \{w_i, w_{n+1}, \ldots, w_b\} \ (i = 1, \ldots, n-1) \\ \{v_n, \ldots, v_m\} & \to & \{w_n, \ldots, w_b\} \\ \{v_{m+1}, \ldots, v_a\} & \to & \{w_{n+1}, \ldots, w_b\}. \end{array}$$

It is easy to check that only kings of A are $v_1, \ldots, v_m$ and only kings of B are $w_1, \ldots, w_n$. Thus, $T(A, B)$ is of the type $(a, m; b, n)$. □

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