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TAYLOR FORMULA OF BOOLEAN AND PSEUDO-BOOLEAN FUNCTION

Koriolan Gilezan
Institute of Mathematics, University of Novi Sad,
21000 Novi Sad, Trg Dositeja Obradovića 4, Yugoslavia

Abstract

Partial derivatives of generalized pseudo-Boolean functions are defined and it is shown that each generalized pseudo-Boolean function can be represented by these partial derivatives.

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1. Partial derivatives of Boolean functions

Partial derivatives of Boolean functions are defined in [8]. Let us recall these notions in order to fix the notations.

Let $(B, \cup, \cdot, ', 0, I)$ be a Boolean algebra, where $x + y = x' \cdot y \cup x \cdot y'$ for $x, y \in B$. A Boolean function is every mapping f from B^n into B, where B^n is the direct product of the set B.

A partial derivative of a Boolean function $f : B^n \to B$ with the variable x_i $(1 \le i \le n)$ is a Boolean function

$$\frac{\partial f}{\partial x_i} : B^{n-1} \to B$$

defined by

$$\frac{\partial f}{\partial x_i}(x_1, ..., x_i, x_{i+1}, ..., x_n) \stackrel{def}{=} f((x_1, ..., x_{i-1}, I, x_{i+1}, ..., x_n)+ \\ + f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n),\ (1 \le i \le n).$$

Partial derivatives of higher order are defined inductively.

The following theorem is proved in [8].

Theorem 1. *The following Taylor formula holds for every Boolean function* $f : B^n \to B$ *: for every* $A \in B^n$,

$$f(x) = f(A) + \sum_{m=1}^{n} \sum_{i_1,...,i_m}^{1,2,...,n} (x_{i_1} + \alpha_{i_1}) \cdot (x_{i_2} + \alpha_{i_2}) ... (x_{i_m} + \alpha_{i_m}) \cdot \\ \cdot \frac{\partial^m f(A)}{\partial x_{i_1} ... \partial x_{i_m}}.$$

(see [8] for more detail).

2. Partial derivatives of lattice functions

Partial derivatives of lattice functions are derived in [3].

Let $(L, \vee, \wedge)$ be a lattice, let $L_1, L_2, ..., L_n$ be finite subsets of L, and let $S = L_1 \times L_2 \times ... \times L_n$ be the direct product. A lattice function is every mapping f from S into L, i.e. $f : S \to L$.

A partial derivative of a lattice function $f : S \to L$ with the variable x_i $(1 \le i \le n)$ is a lattice function

$$\frac{\partial f}{\partial x_i} : S \to L$$

defined by

$$\frac{\partial f}{\partial x_i}(x) \stackrel{def}{=} f(x_1, x_2, ..., x_n) \vee f(x_1, x_2, ..., x_{i \oplus 1}, ..., x_n),\ (1 \le i \le n),$$

$\oplus$ is the addition modulo m_i, m_i is the cardinality of the set L_i $(1 \le i \le n)$.

3. Partial derivatives of ring functions

Partial derivatives of ring functions are defined in [3].

Let $L = \{0, 1, 2, ..., p-1\}$ be a set and let $\oplus$ and $\cdot$ be the addition and the multiplication modulo p, respectively. Then $(L, \oplus, \cdot)$ is a ring. If $L_1, L_2, ..., L_n$ are the subsets of L, then $S = L_1 \times L_2 \times ... \times L_n$ denotes their direct product.

A ring function is any mapping of the set S into L, i.e. $f : S \to L$.

The partial derivative of a ring function $f : S \to L$ with the variable x_i $(1 \leq i \leq n)$ is a ring function

$$\frac{\partial f}{\partial x_i} : S \to L$$

defined by

$$\frac{\partial f}{\partial x_i}(x) \overset{def}{=} f(x_1, ..., x_{i \oplus 1}, ..., x_n) \oplus f'(x_1, ..., x_n),$$

where $f \oplus f' = 0$.

4. Partial derivatives of generalized pseudo-Boolean functions

Let $(P, +, \cdot)$ be a ring such that $\{0, 1\} \subset P$, where 0 is the zero element for the binary operation " + " and 1 is the identity element for the binary operation " $\cdot$ ". Let L be an arbitrary finite set.

A generalized pseudo-Boolean function (GPB function) is every mapping f of the set L^n into P, i.e. $f : L^n \to P$, where L^n is the direct product of L.

Let us introduce the following relation on L

$$[x_i]_{a_i} = \begin{cases} 1, & \text{if } \ x_i = a_i \\ 0, & \text{if } \ x_i \neq a_i, \ x_i, a_i \in L, \end{cases}$$

$[X]_A$ will denote the product $[x_1]_{a_1} \cdot [x_2]_{a_2} \cdot ... \cdot [x_n]_{a_n}$, where $X = (x_1, x_2, ..., x_n)$ and $A = (a_1, a_2, ..., a_n)$, i.e. $A, X \in L^n$. Furthermore, $x + x' = x' + x = 0$, where $x' = -x$, for $x, x' \in P$.

Every GPB function can be represnted in the following form:

$$f(x) = \sum_{A \in L^n} f(A) \cdot [X]_A.$$

Partial derivatives of GPB functions are defined in [4].

Definition 1. *A partial derivative of GPB functions $f : L^n \to P$ with the variable x_i $(1 \leq i \leq n)$ is a GPB function*

$$\frac{\partial f_a}{\partial x_i} : L^n \to P, \quad i = 1, 2, ..., n,$$

defined by

$$\text{(1.1)} \quad \frac{\partial f_a}{\partial x_i}(x) = f(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) - f(x),\ a \in L,\ (1 \leq i \leq n).$$

Partial derivatives of higher order are GPB functions

$$\frac{\partial^m f_{a_{i_1}...a_{i_m}}}{\partial x_{i_1}...\partial x_{i_m}} : L^n \to P, \quad m \geq 1,$$

defined inductively by

$$\text{(1.2)} \quad \frac{\partial^m f_{a_{i_1}...a_{i_m}}}{\partial x_{i_1}...\partial x_{i_m}}(x) = \frac{\partial}{\partial x_{i_m}}(...(\frac{\partial}{\partial x_{i_m}}(\frac{\partial f_{a_1}(x)}{\partial x_{i_1}})a_{i_2})...)a_{i_m},$$

$$1 \leq i \leq n, \quad a_{i_j} \in L, \quad 1 \leq j \leq m.$$

The following properties follows immediately from Definition 1.

If f and g $(f : L^n \to P,\ g : L^n \to P)$ are GPB functions and $c \in P$, then for every $a, b \in L$,

(2.1) f does not contain x_i, $(\forall a \in L)(\frac{\partial f_a}{\partial x_i} = 0), (1 \leq i \leq n)$

$$\text{(2.2)} \quad \frac{\partial (cf)_a}{\partial x_i} = c\frac{\partial f_a}{\partial x_i},\ (1 \leq i \leq n),$$

$$\text{(2.3)} \quad \frac{\partial (f+g)_a}{\partial x_i} = \frac{\partial f_a}{\partial x_i} + \frac{\partial g_a}{\partial x_i},\ (1 \leq i \leq n),$$

$$\frac{\partial(f\cdot g)_a}{\partial x_i}=\frac{\partial f_a}{\partial x_i}g+f\cdot\frac{\partial g_a}{\partial x_i}+\frac{\partial f_a}{\partial x_i}\cdot\frac{\partial g_a}{\partial x_i},(1\le i\le n), \tag{2.4}$$

$$\frac{\partial^2 f_{ab}}{\partial x_i\partial x_j}=\frac{\partial^2 f_{ba}}{\partial x_j\partial x_i},(1\le i\le n,\ 1\le j\le m) \tag{2.5}$$

$$\frac{\partial f\underbrace{aa...a}_{m}}{\partial x_i^m}=(-1)^{m+1}\frac{\partial f_a}{\partial x_i},\ (m\ge 1,\ 1\le i\le m) \tag{2.6}$$

Theorem 2. *Let $f: L^n\to P$ be a GPB function and let $A=(a_1,a_2,...,a_n)$ and $B=(b_1,b_2,...,b_n)$ be two vectors from L^n, then:*

$$f(B)=f(A)+\sum_{m=1}^{n}\sum_{i_1,...,i_m}^{1,...,n}\frac{\partial^m f^{(H)}_{b_{i_1}...b_{i_m}}}{\partial x_{i_1}...\partial x_{i_m}},(i_1<i_2<...<i_m). \tag{3}$$

$$i_1,i_2,...,i_m=1,2,...,n.$$

Proof. For $n=1$ the formula (3) is of the form

$$f(a_1)+\frac{\partial f_{b_1}(a_1)}{\partial x_1}=f(b_1), \tag{3.1}$$

where by Definition 1

$$\frac{\partial f_{b_1}(x_1)}{\partial x_1}=f(b_1)-f(x_1)$$

$$\frac{\partial f_{b_1}(a_1)}{\partial x_1}=f(b_1)-f(a_1).$$

Hence, formula (3.1) is true.

For $n=2$ formula (3) is of the form

$$f(a_1,a_2)+\frac{\partial f_{b_1}(a_1,a_2)}{\partial x_1}+\frac{\partial f_{b_2}(a_1,a_2)}{\partial x_2}=\frac{\partial^2 f_{b_1b_2}(a_1,a_2)}{\partial x_1\partial x_2}=f(b_1,b_2) \tag{3.2}$$

where by Definition 1 (1.1 and 1.2)

$$\frac{\partial f_{b_1}(x_1,x_2)}{\partial x_1}=f(b_1,x_2)-f(x_1,x_2),$$

$$\frac{\partial f_{b_2}(x_1,x_2)}{\partial x_2} = f(b_2,x_1) - f(x_1,x_2),$$

$$\frac{\partial^2 f_{b_1,b_2}(x_1,x_2)}{\partial x_1 \partial x_2} = f(b_1,b_2) - f(x_1,b_2) - f(b_1,x_2) + f(x_1,x_2),$$

(3.2.1.)
$$\frac{\partial f_{c_1}(a_1,a_2)}{\partial x_1} = f(b_1,b_2) - f(a_1,a_2)$$

$$\frac{\partial f_{b_2}(a_1,a_2)}{\partial x_2} = f(a_1,b_2) - f(a_1,a_2)$$

(3.2.2.)
$$\frac{\partial^2 f_{b_1 b_2}(a_1,a_2)}{\partial x_1 \partial x_2} = f(b_1,b_2) - f(a_1,a_2) - f(b_1,a_2) + f(a_1,a_2),$$

By substituting (3.2.1) and (3.2.2) in the left-hand side of (3.2) we obtain that (3.2) is true.

If $A = B$, then by the properties of partial derivatives we obtain that

$$\frac{\partial^m f_{b_{i_1}\dots b_{i_m}}(b_1,\dots,b_n)}{\partial x_i \dots \partial x_{i_m}} = 0, \quad \begin{array}{l}(i_1 < i_2 < \dots < i_m),\\ (i_1,i_2,\dots,i_n = 1,2,\dots,n).\end{array}$$

Therefore $f(B) = f(A)$ for $B = A,\ A, B \in L^n$.

Let substitute $A = (a_1,\dots,a_{n-1},x_n)$ in (3), i.e.

$$f'(x_n) = f(a_1,a_2,\dots,a_{n-1},x_n) + \sum_{m=1}^{n-1} \sum_{i_1,\dots,i_m}^{1,2,\dots,n-1} \frac{\partial^m f_{b_{i_1}\dots b_{i_m}}(a_1,\dots,a_{n-1},x_n)}{\partial x_{i_1} \dots \partial x_{i_m}}$$

$$i_1, i_2 < \dots < i_m.$$

Then, by (3.1) we obtain

(3.3)
$$f'(b_n) = f'(a_n) + \frac{\partial f'_{b_n}(a_n)}{\partial x_n}$$

Finally, if we develope $f'(a_n)$ in (3.3), then the indices $b_{i_1},\dots,b_{i_m}$ of partial derivatives form the following set of subsets

$$P = \{x | x \subset \{b_1,\dots,b_{n-1}\},\ x \neq \emptyset\}.$$

On the other hand, if we develope $\dfrac{\partial f'_{b_n}(a_n)}{\partial x_n}$ in (3.3), then the indices $b_{i_1}, ..., b_{i_m}$ of partial derivatives form the following set of subsets

$$P' = \{X \cup \{b_n\} \mid x \subset \{b_1, ..., b_{n-1}\}\}.$$

Obviously, $P \cap P' = \emptyset$ and

$$P \cup P' = \{X \mid x \subset \{b_1, ..., b_n\},\ x \neq \emptyset\}.$$

Hence, the indices $p_{i_1}, ..., p_{i_m}$ in (3.3) range over the partitive set of $\{b_1, ..., b_m\}$. This completes the proof of Theorem 2. □

Theorem 3. *Every GPB function $f : L^n \to P$ satisfies the following (Taylor) formula: for every $A \in L^n$*
(4)

$$f(x) = f(A) + \sum_{m=1}^{n} \sum_{i_1,...,i_n}^{1,2,...,n} \sum_{c_{i_1},...,c_{i_m} \in L} \frac{\partial^m f_{c_{i_1},...,c_{i_m}}(A)}{\partial x_{i_1} ... \partial x_{i_m}} \cdot [x_{i_1}]_{c_{i_1}} \cdot [x_{i_2}]_{c_{i_2}} ... [x_{i_m}]_{c_{i_m}}$$

$$(i_1 < ... < i_m).$$

Proof. Let us take $X = A$ in (4), then the products

(4.1) $\quad [a_{i_1}]_{c_{i_1}} \cdot [a_{i_2}]_{c_{i_2}} ... [a_{i_m}]_{c_{i_m}},\ m = 1, 2, ..., n,\ i_1, i_2, ..., i_m = 1, 2, ..., n.$

are equal 1 if an only if $c_{i_m} = a_{i_m}$ for every m. In the rest of the cases these products are 0. Therefore the partial derivatives in (4) are 0 whenever the products in (4.1) are 1 and vice versa, i.e. if the partial derivatives are different from 0, then the products in (4.1) are 0.

Thus, it follows from (4) that $f(X) = f(A)$ for $X = A$.

If we substitute $X = B \neq A$ in (4), then one of the products

(4.2) $\quad [b_{i_1}]C_{i_1} \cdot [b_{i_2}]_{c_{i_2}} ... [b_{i_m}]_{c_{i_m}},\ m = 1, 2, ..., n;\ i_1, i_2, ..., i_m = 1, 2, ..., n$

is equal to 1 if and only if $c_{i_m} = b_{i_m}$ for every m, while all the others are 0. In this case for $X = B \neq A$ according to (4.2) formula (4) is transformed into formula (3) from Theorem 2.

This completes the proof of Theorem 3. □

Example. Let us take $L = \{0,1,2\}$, a ring $(P, t, \cdot)$ and let $2 = (2,2,2) \in L^3$. According to Theorem 3 the Taylor formula of a GPB function $f : L^3 \to P$ is

$$f(x,y,z) = f(2) + \frac{\partial f_0(2)}{\partial x}[x]_0 + \frac{\partial f_1(2)}{\partial x}[x]_1 + \frac{\partial f_0(2)}{\partial y}[y]_0 + \frac{\partial f_1(2)}{\partial y}[y]_1$$

$$+\frac{\partial f_0(2)}{\partial z}[z]_0 + \frac{\partial f_1(2)}{\partial z}[z]_1 + \frac{\partial^2 f_{00}(2)}{\partial x \partial y}[x]_0 \cdot [y]_0 + \frac{\partial^2 f_{01}(2)}{\partial x \partial y}[x]_0 \cdot [y]_1$$

$$+\frac{\partial^2 f_{10}(2)}{\partial x \partial y}[x]_1 \cdot [y]_0 + \frac{\partial^2 f_{11}(2)}{\partial x \partial y}[x]_1 \cdot [y]_1 + \frac{\partial^2 f_{00}(2)}{\partial x \partial z}[x]_0 \cdot [z]_0$$

$$\frac{\partial^2 f_{01}(2)}{\partial x \partial y}[x]_0 \cdot [y]_1 + \frac{\partial^2 f_{01}(2)}{\partial x \partial z}[x]_0 \cdot [z]_1$$

$$\frac{\partial^2 f_{10}(2)}{\partial x \partial z}[x]_1[z]_0 + \frac{\partial^2 f_{11}(2)}{\partial x \partial z}[x]_1[z]_1 + \frac{\partial^2 f_{00}(2)}{\partial y \partial z}[y]_0 \cdot [z]_0 + \frac{\partial^2 f_{01}(2)}{\partial y \partial z}[y]_0 \cdot [z]_1$$

$$\frac{\partial^2 f_{10}(2)}{\partial y \partial z}[y]_1[z]_0 + \frac{\partial^2 f_{11}(2)}{\partial y \partial z}[y]_1[z]_1 + \frac{\partial^2 f_{000}(2)}{\partial x \partial y \partial z}[x]_0 \cdot [y]_0 \cdot [z]_0$$

$$+\frac{\partial^3 f_{001}(2)}{\partial x \partial y \partial z}[x]_0[y]_0[z]_1 + \frac{\partial^3 f_{010}(2)}{\partial x \partial y \partial z}[x]_0[y]_1[z]_0 + \frac{\partial^3 f_{100}(2)}{\partial x \partial y \partial z}[x]_1[y]_0[z]_1$$

$$+\frac{\partial^3 f_{110}(2)}{\partial x \partial y \partial z}[x]_1[y]_1[z]_0 + \frac{\partial^3 f_{101}(2)}{\partial x \partial y \partial z}[x]_1[y]_0[z]_1 + \frac{\partial^3 f_{011}(2)}{\partial x \partial y \partial z}[x]_0[y]_1[z]_1$$

$$+\frac{\partial^3 f_{111}(2)}{\partial x \partial y \partial z}[x]_1[y]_1[z]_1.$$

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