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THE APPROXIMATION OF LAYER SOLUTION BY LEGENDRE TYPE POLYNOMIALS

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Abstract

The selfadjoint boundary layer problem, described by the second order linear differential equation is considered. The solution is presented as a sum of the reduced solution and layer solution which is approximated by the truncated orthogonal series where certain Legendre-type polynomials were used as the orthogonal basis. The layer solution is constructed upon the layer subinterval determined according to the asymptotic behavior of the exact solution by the use of apropriate resemblance function. This domain decomposition depends on the degree of chosen spectral approximation. The coefficients of the truncated orthogonal series are determined by the collocation method. The upper bound for the error function is constructed and the numerical example is included.

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1. Introduction

We shall consider the selfadjoint boundary layer problem

$$Ly \equiv -\varepsilon^2 y''(x) + g(x)y(x) = f(x)\; x \in [0,1] \tag{1.1}$$

(1.2) $$Gy \equiv (y(0), y(1)) = (A, B)$$

where $\varepsilon > 0$ is a small parameter, $A, B \in R$, $f(x), g(x) \in C^2[0,1]$.

The solution of the problem (1.1), (1.2) represents the stationary state of the evolution equation

$$\frac{\partial y}{\partial t} - \varepsilon^2 \frac{\partial^2 y}{\partial x^2} + \gamma(x,t)y = \varphi(x,t), \; x \in [0,1], t > 0$$

with the conditions

$$y(0,t) = A, y(1,t) = B, t > 0, y(x,0) = y^0(x) \; x \in [0,1],$$

which is the mathematical model for the diffusion-flow problems known in fluid mechanics and heat conduction.

Under the asuumption that

(1.3) $$g(x) \geq K^2 > 0, \; K \in R$$

we know that the problem (1.1), (1.2) is inverse monotone and that it has the unique solution $y(x) \in C^2[0,1]$. We also know that the reduced solution

(1.4) $$y_r(x) = f(x)/g(x)$$

sufficiently well approximates the exact solution out of the boundary layers, which occure at $x = 0$ and $x = 1$. From the asymptotic theory we have that

(1.5) $$|y(x) - y_r(x)| < C_1 e^{-\frac{Kx}{\varepsilon}} + C_2 e^{-\frac{K(1-x)}{\varepsilon}} + C_3 \varepsilon^2 \; x \in (0,1)$$

and we can see that the layer length is of order $0(\varepsilon)$.

The problem (1.1), (1.2) has already been investigated by the author in a few of her papers, see e.g. [1]. In those researchies the modification of standard spectral approximation was used, after the determination of numerical layer length. Here that procedure will be altered to the special domain decomposition, and instead of classical orthogonal polynomials, the Legendre-type polynomials of special kind will be used as the orthogonal basis.

Legendre-type polynomials represent a polynomial set which satisfy fourth order differential equation

$$(1.6)\qquad (t^2-1)^2 y^{iv} + 8t(t^2-1)y''' + 16(t^2-1)y'' + \\ + 8ty' - n(n+1)(n^2+n+2)y = 0$$

and they were introduced by H.L. Krall in 1940. In his paper [4] A. Krall investigated some of their properties. Denoting them by $P_n(t)$, he has shown that they satisfy three term recurrence relation

$$(1.7)\qquad P_{n+1}(t) = \frac{(2n+1)(n^2+n+2)}{(n+1)(n^2-n+2)} tP_n(t) - \\ \frac{n(n^2+3n+4)}{(n+1)(n^2-n+2)} P_{n-1}(t), P_0(t) = 1, P_{-1}(t) = 0$$

and that

$$(1.8)\qquad P_n(1) = 1, P_n(-1) = (-1)^n \; \forall n \in N.$$

The author has already used similar polynomial set in her paper [2] to the modified spectral solution for nonselfadjoint problem.

2. Domain decomposition

We are going to look for the solution of the problem (1.1), (1.2) in the form

$$(2.1)\qquad y(x) = y_r(x) + y_\varepsilon(x),$$

where $y_\varepsilon(x)$ satisfies

$$(2.2)\quad Ly_\varepsilon = \varepsilon^2 y_r''(x)\; Gy_\varepsilon = (A^0, B^0), A^0 = A - y_r(0), B^0 = B - y_r(1).$$

The first step is to approximate function $y_\varepsilon(x)$ by

$$(2.3)\qquad u(x) = \begin{cases} v(x) & x \in [0, c\varepsilon] \\ 0 & x \in [c\varepsilon, 1 - c_1\varepsilon], \quad c, c_1 \in R^+ \\ w(x) & x \in [1 - c_1\varepsilon, 1] \end{cases}.$$

The constant c (and similarly c_1), which performs the domain decomposition will be determined in such a way that it depends on the degree n of the truncated orthogonal series which will approximate the layer solutions. In that purpose we use the resemblance function defined by the author in [1]

$$(2.4)\qquad p(x) = A^0(1 - \frac{x}{c\varepsilon})^n.$$

The constant c is going to be evaluated from the request that $p(x)$ satisfies the differential equation in (2.2) at the layer point $x = 0$. This gives

Lemma 1. *The constant c is*

$$c = \sqrt{\frac{A^0 n(n-1)}{A^0 g(0) - \varepsilon^2 y_r''(0)}} \approx \sqrt{\frac{n(n-1)}{g(0)}}. \tag{2.5}$$

Proof. If we introduce (2.4) into differential equation $Ly_\varepsilon = \varepsilon^2 y_r''(x)$, for $x = 0$ we come to the equation

$$-n(n-1)A^0 + c^2 g(0) A^0 = c^2 \varepsilon^2 y_r''(0)$$

and its positive solution is given by (2.5). When ε is sufficiently small, the constant c may be determined as

$$c = \sqrt{\frac{n(n-1)}{g(0)}}. \tag{2.6}$$

In the similar way we have that

$$c_1 = \sqrt{\frac{n(n-1)}{g(1)}}. \tag{2.7}$$

3. Legendre-type approximation

In purpose to construct the spectral approximation to (2.3) we first have to remark that the function $v(x)$ satisfies the boundary value problem

$$Lv = \varepsilon^2 y_r''(x),\ x \in [0, c\varepsilon],\ v(0) = A^0,\ v(c\varepsilon) = 0. \tag{3.1}$$

This problem is still singularly perturbed. If we introduce the stretching variable

$$x = \frac{1}{2} c\varepsilon(t+1) \tag{3.2}$$

we shall obtain a non-perturbed problem

$$L_t w \equiv -4w''(t) + c^2 G(t) w(t) = c^2 \varepsilon^2 Y_r(t),\ t \in [-1, 1] \tag{3.3}$$

(3.4) $$w(-1) = A^0,\ w(1) = 0,$$

where

$$w(t) = v(\frac{1}{2}c\varepsilon(t+1)),\ G(t) = g(\frac{1}{2}c\varepsilon(t+1)),\ Y_r(t) = y_r''(\frac{1}{2}c\varepsilon(t+1)).$$

We are going to construct the spectral approximation , due to Legendre-type polynomials, for the problem (3.3), (3.4) in the form

(3.5) $$w_n(t) = \sum_{k=0}^{n} a_k P_k(t).$$

This will give us the following

Theorem 1. *The coefficients a_k, $k = 0, \ldots, n$ of the truncated series (3.5) represent the solution of the system*

(3.6) $$\sum_{k=0}^{n} p_{k,i} a_k = b_i,\ i = 0, \ldots, n$$

with

(3.7) $$p_{k,0} = 1, p_{k,n} = (-1)^k,\ k = 0, \ldots, n$$

(3.8) $$p_{k,i} = -4P_k''(t_i) + c^2 G(t_i) P_k(t_i),\ b_i = c^2\varepsilon^2 Y_r(t_i),\ t_i = \cos\frac{i\pi}{n},$$
$$k = 0, \ldots, n, \quad i = 1, \ldots, n-1.$$

Proof. We substitute (3.5) into (3.3) and ask that the obtained equality is satisfied at Gauss-Lobatto nodes

$$t_i = \cos\frac{i\pi}{n},\ i = 1, \ldots, n-1.$$

This gives equations in (3.6) with the coefficients (3.8). The first and the last equation, determined by (3.7) is obtained directly by substituting (3.5) into (3.4) and using the property (1.8).

Remark 1: The values for $P_k(t_i)$ and $P_k''(t_i)$ can be evaluated successively by the use of (1.7).

Remark 2: The procedure for the construction of the spectral approximation of the layer solution at the point $x = 1$ is analogue.

4. The error estimate

It is obvious from (2.3) that out of the boundary layers the exact solution is approximated by the reduced solution $y_r(x)$. Using the proposed domain decomposition we can prove the following

Theorem 2. *For $x \in [c\varepsilon, 1 - c_1\varepsilon]$, where c and c_1 are given by (2.6) and (2.7) it holds that*

$$|y(x) - y_r(x)| < C(\varepsilon^2 + e^{-\frac{K(n-1)}{g}}), \tag{4.1}$$

where K is given by (1.3) and

$$g = \max(\sqrt{g(0)}, \sqrt{g(1)}). \tag{4.2}$$

Proof. From (2.6) (and similarly (2.7)) we can see that for $x > c\varepsilon$ we have that

$$e^{-\frac{Kx}{\varepsilon}} < e^{-Kc} \leq e^{-\frac{K(n-1)}{g}},$$

because of

$$c > \frac{n-1}{\sqrt{g(0)}} \geq \frac{n-1}{g}.$$

This, together with (1.5) gives us (4.1).

For the error estimate inside the layer we can apply the result of

Lemma 2. *Let $K_1^2 = \min g(x)$, $K_2^2 = \max g(x)$ for $x \in [0, c\varepsilon]$. For ε sufficiently small it holds that*

$$v_2(x) < v(x) < v_1(x),\ v_i(x) = A^0 \cdot \frac{shK_i(c - \frac{x}{\varepsilon})}{shK_i c} i = 1, 2. \tag{4.3}$$

This lemma was proved in [1].

Now we can prove

Theorem 3. *For $x \in [0, c\varepsilon]$, where c is given by (2.6) we have that*

$$|y(x) - y_n(x)| < \max_i\{|v_i(x) - u_n(x)|\} + C(\varepsilon^2 + e^{-\frac{K(n-1)}{g}}) = d(x) \tag{4.4}$$

where

$$y_n(x) = y_r(x) + u_n(x),\ u_n(x) = w_n(\frac{2x}{c\varepsilon} - 1). \tag{4.5}$$

Proof. According to (2.1), (2.3) and (4.5) we can see that

$$|y(x) - y_n(x)| = |y_\varepsilon(x) - u_n(x)| \leq |y_\varepsilon(x) - v(x)| + |w(t) - w_n(t)|.$$

By the use of the principle of the inverse monotonicity to the problem

$$Ly_\varepsilon = \varepsilon^2 y_r''(x),\ y_\varepsilon(0) = A^0,\ y_\varepsilon(c\varepsilon) = y(c\varepsilon) - y_r(c\varepsilon)$$

and the problem (3.1) we easily come to the conclusion that

$$|y_\varepsilon(x) - v(x)| \leq |y_\varepsilon(c\varepsilon) - v(c\varepsilon)|,$$

and by the use of (4.1) this gives

$$|y_\varepsilon(x) - v(x)| < C(\varepsilon^2 + e^{-\frac{K(n-1)}{g}}),\ x \in [0, c\varepsilon]. \tag{4.6}$$

As for the second term, using the substitution (3.2), after subtracting $w_n(t)$ in (4.3) and taking the maximum of the obtained differences, we come to

$$|w(t) - w_n(t)| \leq \max_i\{|v_i(x) - u_n(x)|\}. \tag{4.7}$$

5. Numerical results

We shall use the following test example from [3]

$$-\varepsilon^2 y'' + \frac{1-\varepsilon}{(2-x)^2} y(x) = \frac{(1-\varepsilon)(x-1)}{(2-x)^2},$$
$$y(0) = 0, \quad y(1) = 0.$$

The exact solution is

$$y(x) = \frac{1 - (2-x)^{-\frac{2}{\varepsilon+1}}}{1 - 2^{-\frac{2}{\varepsilon+1}}} \cdot (1 - \frac{x}{2})^{\frac{1}{\varepsilon}} + x - 1.$$

We have only one boundary layer at $x = 0$. The following tables give the exact error and the error estimate in several points from the boundary layer:

$\varepsilon = 10^{-5}$		$n = 4$		$n = 12$	
x	y	$\|y(x) - y_n(x)\|$	$d(x)$	$\|y(x) - y_n(x)\|$	$d(x)$
0,000001	−0,05	$6.1 \cdot 10^{-4}$	$6.0 \cdot 10^{-4}$	$9.9 \cdot 10^{-6}$	$5.0 \cdot 10^{-5}$
0,000006	−0,26	$2.5 \cdot 10^{-4}$	$2.1 \cdot 10^{-3}$	$8.5 \cdot 10^{-7}$	$3.2 \cdot 10^{-5}$
0,00002	−0,63	$7.1 \cdot 10^{-3}$	$7.2 \cdot 10^{-3}$	$7.4 \cdot 10^{-6}$	$4.2 \cdot 10^{-5}$
0,00004	−0,86	$5.6 \cdot 10^{-3}$	$4.3 \cdot 10^{-3}$	$2.2 \cdot 10^{-6}$	$2.3 \cdot 10^{-5}$

Table 1.

$\varepsilon = 10^{-7}$		$n = 4$		$n = 12$	
x	y	$\|y(x) - y_n(x)\|$	$d(x)$	$\|y(x) - y_n(x)\|$	$d(x)$
0,00000001	−0,05	$2.6 \cdot 10^{-4}$	$6.1 \cdot 10^{-4}$	$8.7 \cdot 10^{-4}$	$8.8 \cdot 10^{-4}$
0,00000006	−0,26	$2.9 \cdot 10^{-4}$	$4.6 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$
0,0000002	−0,63	$7.2 \cdot 10^{-3}$	$7.2 \cdot 10^{-3}$	$4.5 \cdot 10^{-4}$	$6.2 \cdot 10^{-4}$
0,0000004	−0,86	$2.8 \cdot 10^{-3}$	$4.3 \cdot 10^{-3}$	$1.6 \cdot 10^{-5}$	$1.0 \cdot 10^{-5}$

Table 2.

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