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ON n-GROUPS WITH $\{i,j\}$-NEUTRAL OPERATION FOR $\{i,j\} \neq \{1,n\}$

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Abstract

The notion of an $\{i,j\}$-neutral operation of an n-grupoid has been introduced in [5], as a generalization of the neutral element in a grupoid (:1.2). Every n-group (:[1], 1.3), $n \in |N \backslash \{1\}$, has (uniquely determined :1.2.2) $\{1,n\}$-neutral operation (:[5], 1.3.2). The condition $\{i,j\} \neq \{1,n\}$ is fulfilled for $n \geq 3$; for $n = 2$ the equality holds. In the present article, is, among others, given a necessary and sufficient condition for an n-group ($n \geq 3$) to have an $\{i,j\}$-neutral operation with the condition $\{i,j\} \neq \{1,n\}$ (:Theorem 2.1).

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1. Preliminaries

1.1. About the expression a_p^q

Let $p \in \mathbf{N}$, $q \in \mathbf{N} \cup \{0\}$ and let a be a mapping of the set $\{i|\ i \in \mathbf{N} \wedge i \geq p \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence } (= \emptyset); & p > q. \end{cases}$$

For example:
$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1})$, $j \in \{1,\ldots,n\}$, $n \in \mathbf{N}\backslash\{1,2\}$, for $j = n$ stands for

$$A(a_1,\ldots,a_{n-1},A(a_n,\ldots,a_{2n-1})).$$

Besides, in some situations
1) **instead of a_p^q we write** $(a_i)_{i=p}^q$ (briefly: $(a_i)_p^q$); or
2) **instead of a_p^q we write** $\overline{a_i|}_{i=p}^q$ (briefly: $\overline{a_i|}_p^q$).

For example:
a) $(\forall x_i \in Q)_1^q$ for $q > 1$ stands for

$$\forall x_1 \in Q \ldots \forall x_q \in Q\ ^1,$$

for $q = 1$ it stands for

$$\forall x_1 \in Q\ ^2,$$

and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.
b) $\overline{\varphi^{i-1}(b_i)|}_{i=p}^q$ for $q > p$ stands for

$$\varphi^{p-1}(b_p),\ldots,\varphi^{q-1}(b_q),$$

for $q = p$ it stands for

$$\varphi^{p-1}(b_p),$$

and for $q < p$ it stands for an empty sequence $(= \emptyset)$.

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

If a_p^q is a sequence over a set S, $p \leq q$ and the equalities $a_p = \ldots = a_q = b$ $(\in S)$ are satisfied, then

$$a_p^q \quad \textbf{is denoted by} \quad \overset{q-p+1}{b}\ .$$

In connection with this, if $q - p + 1 = r$ (when we assume that there would be no missunderstanding),

$$\textbf{instead of} \quad \overset{q-p+1}{b} \quad \textbf{we write}\ \overset{r}{b}.$$

In addition, we denote **the empty sequence over S** with $\overset{0}{b}$, where b is an arbitrary element from S.

[1]usually, we write: $(\forall x_1 \in Q)\ldots(\forall x_q \in Q)$.
[2]usually, we write: $(\forall x_1 \in Q)$.

1.2. On $\{i,j\}$-neutral operations in an n-groupoid, $n \in \mathbf{N}\backslash\{1\}$

The notion of an $\{i,j\}$-neutral operation in an n-groupoid has been introduced in [5]:

1.2.1: *Let $n \in \mathbf{N}\backslash\{1\}$, $(i,j) \in \{1,\ldots,n\}^2$ and $i < j$. Let also (Q,A) be an n-groupoid and* E *the mapping of the set Q^{n-2} into the set Q. Then,* we say *that* E *is an $\{i,j\}$-neutral [$\{j,i\}$-neutral] operation of (Q,A) iff the following formula holds:*

$$(\forall a_t \in Q)_1^{n-2}(\forall x \in Q)(A(a_1^{i-1}, \mathsf{E}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x \wedge$$

$$\wedge\, A(a_1^{i-1}, x, a_i^{j-2}, \mathsf{E}(a_1^{n-2}), a_{j-1}^{n-2} = x).$$

1.2.2: *The $\{i,j\}$-neutral operation is a generalization of the neutral element of a groupoid. Namely, for $n = 2$, $\mathsf{E}(a_1^{n-2})$ [$= \mathsf{E}(\emptyset)$] is the neutral element of the groupoid (Q,A) (:$n-2=0$, $i=1$, $j=2$; 1.1, 1.2.1). Moreover,* E *is a nullary operation in Q.*

The following proposition holds:

1.2.3 [5]: *Let n be an arbitrary element of the set $\mathbf{N}\backslash\{1\}$ and (Q,A) an arbitraty n-groupoid. Let also $\{i,j\}$ be an arbitrary set such that $(i,j) \in \{1,\ldots,n\}^2$ and $i < j$. Then, (Q,A) has at most one $\{i,j\}$-neutral operation.*

1.3. About n-groups

The notion of an n-group has been introduced in [1]:

1.3.1: *Let $n \in \mathbf{N}\backslash\{1\}$ and let A be a mapping of the set Q^n into the set Q. (Q,A) is said to be an n-semigroup iff for every $i \in \{2,\ldots,n\}$ and for all $x_1^{2n-1} \in Q$ the following equality holds:*

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

(Q,A) is an n-quasigroup iff for every $i \in \{1,\ldots,n\}$ and for all $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds. (Q, A) *is said to be Dörnte n-group (briefly: an n-group) iff* (Q, A) *is both an n-semigroup and an n-quasigroup.*

1.3.2: *Dörnte's n-group is, for* $n = 2$, *a group.*[3]

The following proposition holds:

1.3.3 [5]: *In every n-group* $(n \in \mathbf{N}\backslash\{1\})$ *there is a* $\{1, n\}$*-neutral operation.*

1.4. On Hosszú-Glushkin algebras

The notion of a Hosszú-Glushkin algebra of order n (briefly: nHG-algebra) has been introduced in [7]:

1.4.1: *Let* $\cdot$ *be a binary and* φ *a unary operation in* Q. *Let also* b *be a (fixed) element of the set* Q, *and* n *a (fixed) element of the set* $\mathbf{N}\backslash\{1, 2\}$. $(Q, \{\cdot, \varphi, b\})$ *is said to be a* **Hosszú-Glushkin algebra of order** n *(briefly: nHG-algbra) iff the following hold:*

(1) $(Q, \cdot)$ *is a group;*

(2) $\varphi \in Aut(Q, \cdot)$;

(3) $\varphi^{n-1}(x) \cdot b = b \cdot x$ *for every* $x \in Q$; *and*

(4) $\varphi(b) = b$.

Theorem 1.4.2. (*Hosszú-Glushkin* [2-3])*Let* (Q, A) *be an n-group and* $n \in \mathbf{N}\backslash\{1, 2\}$. *Then, there is an nHG-algebra* $(Q, \{\cdot, \varphi, b\})$ *such that for each* $x_1^n \in Q$ *the equality*

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \tag{5}$$

holds.

By a simple verification we conclude that the following proposition also holds:

[3]Menger's n-quasigroup is, for $n = 2$, also a group (for example [4]).

1.4.3: *Let* $(Q,\{\cdot,\varphi,b\})$ *nHG-algebra* $(n \in \mathbb{N}\backslash\{1,2\})$. *Let also*

$$A(x_1^n) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$$

for all $x_1^n \in Q$. *Then* (Q,A) *is an n-group.*

1.4.4: *We shall say that an nHG-algebra* $(Q,\{\cdot,\varphi,b\})$ **corresponds** *to the n-group* (Q,A) *iff the equality (5) holds for all* $x_1^n \in Q$.

An immediate consequence of Definition 1.4.1 is the following proposition:

1.4.5: *Let* $(Q,\{\cdot,\varphi,b\})$ *be an nHG-algebra. Now, if* $(Q,\cdot)$ *is a commutative group, then* $\varphi^{n-1} = I$, *where* I *is the identity permutation of the set* Q.

2. Main result

Theorem 2.1. *Let* $n \in \mathbb{N}\backslash\{1,2\}$, $(i,j) \in \{1,\ldots,n\}^2$, $i < j$ *and* $\{i,j\} \neq \{1,n\}$. *Let also* (Q,A) *be an n-group and* $(Q,\{\cdot,\varphi,b\})$ **a corresponding** *(arbitrary) nHG-algebra. Then, the following statements are equivalent:*

(i) (Q,A) **has** *an* $\{i,j\}$*-neutral operation; and*

(ii) $(Q,\cdot)$ *is a commutative group,* $\varphi^{i-1} = I$ *and* $\varphi^{j-1} = I$ *(where* I *is the identity permutation of the set* Q*).*

Proof.

1) Let $(Q,\cdot)$ be a group (semigroup) with the neutral element e. In the proof we use the following convention:

$$(1) \qquad \prod_{t=p}^{q} c_t \text{ stands for } \begin{cases} c_p \cdot \ldots \cdot c_q; & p < q \\ c_p; & p = q \\ e; & c_p^q = \emptyset. \end{cases}$$ [4] [5]

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[4] 1.1.

[5] The case $p \leq q$ is based on the following convention: $\prod_{t=1}^{n+1} c_t \overset{def}{=} (\prod_{t=1}^{n} c_t) \cdot c_{n+1}$ and $\prod_{t=m}^{m} c_t \overset{def}{=} c_m$.

For example:

1* $$\prod_{t=n+1}^{n} \varphi^{t-1}(b_{t-2}) = e \ \ (: \overline{\varphi^{t-1}(b_{t-2})}|_{t=n+1}^{n} = \emptyset;\ 1.1,\ (1)).$$

2* $$A(b_1^{i-1}, x, b_i^{j-2}, y, b_{j-1}^{n-2}) =$$

$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \varphi^{i-1}(x) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \varphi^{j-1}(y) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b$$

(:2.1, 1.4.4, (1)).

2) (i) ⇒ (ii):

Let (Q, A) be an n-group ($n \in \mathbf{N}\backslash\{1,2\}$), $(Q, \{\cdot, \varphi, b\})$ **a corresponding** nHG-algebra, e the neutral element of the group $(Q, \cdot)$ and $^{-1}$ the inversing operation in $(Q, \cdot)$. Let also E be an $\{i, j\}$-neutral operation of (Q, A) (:1.2.1) such that the condition

(2) $$\{i, j\} \neq \{1, 2\}$$

holds.

Then, using Hosszú-Glushkin Theorem (:1.4.2) and the convention (1), we conclude that for every $x, b_1^{n-2} \in Q$ the following equalities hold:

(3)
$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \varphi^{i-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \varphi^{j-1}(x) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x$$

and

(4)
$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \varphi^{i-1}(x) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \varphi^{j-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x,$$

hence, we conclude that for every $x, b_1^{n-2} \in Q$ the following equality holds

(5)
$$\varphi^{i-1}(\mathsf{E}(b_1^{n-2})) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \varphi^{j-1}(x) = \varphi^{i-1}(x) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \varphi^{j-1}(\mathsf{E}(b_1^{n-2})).$$

Substituing x by e, we deduce from (3) that for every sequence b_1^{n-2} over Q the following equality holds:
(6)

$$\varphi^{i-1}(\mathsf{E}(b_1^{n-2})) = (\prod_{t=1}^{i-1} \varphi^{t-1}(b_t))^{-1} \cdot b^{-1} \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2}))^{-1} \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t))^{-1}$$

(:1.4.1), hence, since $(Q,\cdot)$ is a group and φ and $^{-1}$ are permutations of the set Q, we conclude that the following proposition holds:

$\hat{1}$*: E is a permutation of the set Q for $n = 3$, and (Q, E) is an $(n-2)$-quasigroup for $n \geq 4$.*

The consequence of the condition (2) is the following proposition:

$\hat{2}$*: If $\overline{\varphi^t(b_t)}|_{t=i}^{j-2}$ is not the empty sequence, that at least one of the variables b_1 and b_{n-2} (for $n = 3$: $b_1 = b_{n-2}$) is not the variable in the term* $\prod_{t=i}^{j-2} \varphi^t(b_t)$.

By Propositions $\hat{1}$, $\hat{2}$ and by the statement connected with (5), we conclude that the following proposition holds:

$\hat{3}$*: $(Q,\cdot)$ is a commutative group.*

Since $\varphi(b^{-1}) = b^{-1}$ (:$\varphi(b) = b$; 1.4.1), if we put in (6) $b_1 = \ldots = b_{n-2} = e$, we conclude that the following proposition holds:

$\hat{4}$: $\mathsf{E}(\overset{n-2}{e}) = b^{-1}$.

By Proposition $\hat{4}$, putting in (3) and (4) $b_1 = \ldots = b_{n-2} = e$, we conclude that also the following proposition holds:

$\hat{5}$: $\varphi^{i-1} = I$ *and* $\varphi^{j-1} = I$.

By Propositions $\hat{3}$ and $\hat{5}$, we finally conclude that the implication (i) $\Rightarrow$ (ii) holds.

3) (ii) $\Rightarrow$ (i):

Let (Q, A) be an n-group ($n \in \mathbf{N}\backslash\{1,2\}$), $(Q, \{\cdot, \varphi, b\})$ **a corresponding** nHG-algebra and let the statement (ii) holds. Let also, for every $b_1^{n-2} \in Q$

$$\mathsf{E}(b_1^{n-2}) \overset{def}{=} (\prod_{t=1}^{i-1} \varphi^{t-1}(b_t))^{-1} \cdot b^{-1} \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2}))^{-1} \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t))^{-1}.$$

Then, for every $x, b_1^{n-2} \in Q$, the following two sequences of equalities hold:

$$A(b_1^{i-1}, \mathsf{E}(b_1^{n-2}), b_i^{j-2}, x, b_{j-1}^{n-2}) =$$

$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot \mathsf{E}(b_1^{n-2}) \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot x \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x; \text{ and}$$

$$A(b_1^{i-1}, x, b_i^{j-2}, \mathsf{E}(b_1^{n-2}), b_{j-1}^{n-2}) =$$

$$(\prod_{t=1}^{i-1} \varphi^{t-1}(b_t)) \cdot x \cdot (\prod_{t=i}^{j-2} \varphi^{t}(b_t)) \cdot \mathsf{E}(b_1^{n-2}) \cdot (\prod_{t=j+1}^{n} \varphi^{t-1}(b_{t-2})) \cdot b = x.$$

Hence, the implication (ii) $\Rightarrow$ (i) also holds.

□

3. Three propositions and an example

If $(Q, \cdot)$ is a **noncommutative** group and $A(x_1^n) \stackrel{def}{=} x_1 \cdot \ldots \cdot x_n$, $n \in \mathbf{N}\backslash\{1,2\}$, then (Q, A) is an n-group **without** $\{i,j\}$-neutral operations with the condition $\{i,j\} \neq \{1,n\}$ (:Theorem 2.1; $\neg$ (i) $\Leftrightarrow$ $\neg$ (ii)). Besides, for example, if $(Q, \cdot)$ is a commutative group in which not every $a \in Q$ is selfinverse, $^{-1}$ is the inversing operation in $(Q, \cdot)$ and $A(x_1^3) \stackrel{def}{=} x_1 \cdot x_2^{-1} \cdot x_3$, then (Q, A) is a 3-group **without** $\{i,j\}$-neutral operations with the condition $\{i,j\} \neq \{1,n\}$ (:Theorem 2.1; $\varphi = ^{-1}$). Hence, the following proposition holds:

Proposition 3.1. *There exist n-groups ($n \in \mathbf{N}\backslash\{1,2\}$) without $\{i,j\}$-neutral operations with the condition $\{i,j\} \neq \{1,n\}$.*

Theorem 3.2. *Let $n \in \mathbf{N}\backslash\{1,2\}$, (Q, A) an n-group, and* **e** *its $\{1,n\}$-neutral operation (:1.2.1, 1.2.3, 1.3.3). Then the following statements are equivalent:*

(i) (Q, A) is a commutative n-group;

(ii) **e** *is an $\{i,j\}$-neutral operation of the n-group for every $(i,j) \in \{(p,q)|$*

$(p,q) \in \{1,\ldots,n\}^2 \wedge p < q\}$;

(iii) (Q, A) **has** *an* $\{1, n-1\}$*-neutral operation; and*

(iv) (Q, A) **has** *a* $\{2, n\}$*-neutral operation.*

Proof.

1) (i) $\Rightarrow$ (ii):

(Q, A) is commutative iff for every permutation α of the set $\{1, \ldots, n\}$ and for every $x_1^n \in Q$ the following equality holds

$$A(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = A(x_1^n).$$

Hence, by Proposition 1.3.3, we conclude that the implication (i) $\Rightarrow$ (ii) holds.

2) (ii) $\Rightarrow$ (iii):

By (ii), **e** is also an $\{1, n-1\}$-neutral operation of the n-group (Q, A).

3) (iii) $\Rightarrow$ (iv):

Let $(Q, \{\cdot, \varphi, b\})$ be an nHG-algebra **corresponding** to the n-group (Q, A) (:1.4.1, 1.4.4). Hence, by Theorem 2.1 and by the condition (iii), we conclude that

a) $(Q, \cdot)$ is a commutative group; and

b) $\varphi^{n-2} = I$.

Further, by a) and by Proposition 1.4.5, we conclude that the following equality holds:

c) $\varphi^{n-1} = I$.

From b) and c) it follows that:

d) $\varphi = I$.

Finally, by a), c) and d), and by Theorem 2.1, we cocnlude that (Q, A) has a $\{2, n\}$-neutral operation.

4) (iv) $\Rightarrow$ (i):

Let $(Q, \{\cdot, \varphi, b\})$ be an nHG-algebra **corresponding** to the n-group (Q, A) (:1.4.1, 1.4.4). Thereby, and also by Theorem 2.1 and condition (iv), we conclude that $(Q, \cdot)$ is a commutative group and that $\varphi = I$, hence, by 1.4.4, we finally conclude that (Q, A) is a commutative n-group.

□

The consequence of Theorem 3.2 is the following proposition:

Proposition 3.3. *Let $n \in \{3,4\}$ and (Q,A) be an n-group. Then the following statements are equivalent:*

(i) (Q,A) **has** *an $\{i,j\}$-neutral operation with the condition $\{i,j\} \neq \{1,n\}$; and*

(ii) (Q,A) is a commutative n-group.

Example 3.4. Let $(Q,\cdot)$ be a commutative group in which not every $a \in Q$ is selfinverse, e its neutral element and $^{-1}$ it the inversing operation. Let also

$$A(x_1^5) \overset{def}{=} x_1 \cdot x_2^{-1} \cdot x_3 \cdot x_4^{-1} \cdot x_5 .$$

Then:

a) (Q,A) is a 5-group (:1.4.3; $\varphi = ^{-1}$, $b = e$);

b) for

$$\mathbf{e}(a_1^3) \overset{def}{=} a_1 \cdot a_2^{-1} \cdot a_3,$$

$\mathbf{e}$ is an $\{1,5\}$-neutral operation of the 5-group (Q,A);

c) for

$$\mathsf{E}_1(a_1^3) \overset{def}{=} a_1^{-1} \cdot a_2 \cdot a_3,$$

E_1 is a $\{3,5\}$-neutral operation of the 5-group (Q,A); and

d) for

$$\mathsf{E}_2(a_1^3) \overset{def}{=} a_1 \cdot a_2 \cdot a_3^{-1},$$

E_2 is an $\{1,3\}$-neutral operation of the 5-group (Q,A).

4. Remark

Groups can be considered as universal algebras in different ways. For example:

4.1: An algebra $(Q,\{\cdot,^{-1}\})$, where $\cdot$ is a binary and $^{-1}$ an unary operation on Q, is considered to be a group iff the following laws are fulfilled:

$$\begin{aligned} &(x \cdot y) \cdot z = x \cdot (y \cdot z); \\ &a^{-1} \cdot (a \cdot x) = x; \text{ and} \\ &(x \cdot a) \cdot a^{-1} = x. \end{aligned}$$

An analogous treatement of n-groups (and in this case for every $n \in \mathbf{N}\backslash\{1\}$) is based on [6]. Namely the following proposition holds:

4.2 [6]: *Let (Q, A) be an n-semigroup and $n \in \mathbf{N}\backslash\{1\}$. Then:*

(i) There is at most one $(n-1)$-ary operation $^{-1}$ in Q such that the following formulas hold

(1)
$$(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$$ [6]

and

(2) $$(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x;$$

(ii) If there is an $(n-1)$-ary operation $^{-1}$ in Q such that the formulas (1) and (2) are satisfied, then (Q, A) is an n-group; and

(iii) If (Q, A) is an n-group, then there is an $(n-1)$-ary operation $^{-1}$ in Q such that the formulas (1) and (2) hold.[7]

Besides, n-groups for $n \geq 3$ can be considered as universal algebras $(Q, \{A, \mathbf{e}\})$, where (Q, A) is an n-semigroup and $\mathbf{e}$ is an $\{1, n\}$-neutral operation of the n-groupoid (Q, A) (:1.2.1, 1.2.3, 1.3.3). This is based on the following proposition:

4.3 [5]: *For $n \geq 3$, an n-semigroup (Q,A) is an n-group iff (Q,A) has a $\{1, n\}$-neutral operation.*

Besides, the following proposition, for example, also holds:

[6] In [6], instead of $^{-1}$, f has been used, and instead $(a_1^{n-2}, a)^{-1}$, $f(a_1^{n-2}, a)$ has been written.

[7] $(a_1^{n-2}, a)^{-1} \overset{def}{=} f\overset{(2)}{\mathbf{e}}(a_1^{n-2}, a, a_1^{n-2})$, where $\overset{(2)}{\mathbf{e}}$ is a $\{1, 2n-1\}$-neutral operation of a $(2n-1)$-group $(Q, \overset{2}{A})$; $\overset{2}{A} \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$. We note that for $n = 2$, this is the inversing operation in a group.

4.4 [6]: *Let $n \in \mathbf{N}\backslash\{1\}$. Let also (Q,A) be an n-group,* $\mathbf{e}$ *its* $\{1,n\}$*-neutral operation and* $^{-1}$ *the inversing operation in (Q,A) (:4.2). Then the following formula holds:*

$$(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)\left(A((a_1^{n-2},a)^{-1},a_1^{n-2},a) = \mathbf{e}(a_1^{n-2})\wedge\right.$$
$$\left.\wedge A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}) = \mathbf{e}(a_1^{n-2})\right).$$

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