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MSOR METHOD FOR SOLVING LINEAR INTERVAL EQUATIONS

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Abstract

In this paper we consider modified overrelaxation (MSOR) method for solving linear interval equations. We give some sufficient conditions for the convergence of interval MSOR method for some classes of interval matrices.

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1. Introduction

Young [7] presented the MSOR method and showed that some variants of this method were faster than the SOR method.

In this paper we will consider the interval version of the MSOR method and we will establish some sufficient conditions of convergence for this method.

Let us consider the linear interval equations

$$Ax = b, \quad \text{where} \quad A \in IR^{n,n}, 0 \notin A_{ii}, i \in N, x, b \in IR^{n,1}.$$

Let $A = \begin{bmatrix} D_1 & T_1 \\ S_1 & D_2 \end{bmatrix}$ and $w, w' \in R, w \neq 0$, where D_1 and D_2 are square interval nonsingular diagonal matrices of order k and $n-k$, respectively.

In order to approximate the solution of $Ax = b$, we have the MSOR method given by

$$x^{(i+1)} = M_{w,w'}x^{(i)} + d_{w,w'}, \; i = 0, 1, \ldots \tag{1.1}$$

with

$$M_{w,w'} = (D - w'S)^{-1}[(1-w)D + (w-w')R + wT]$$

where $b = [b_1, b_2]^T$ is partitioned in accordance with the partioning of A,

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ -S_1 & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & -T_1 \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix} \quad \text{and}$$

$$d_{w,w'} = w(D - w'S)^{-1} \begin{bmatrix} -D_1^{-1} & b_1 \\ -D_2^{-1} & b_2 \end{bmatrix}.$$

In the seguel we need some definitions and results presented in [4].

Let us first consider the following notations: $IR^{n,n}$ is the set of all interval matrices

$$X := [\underline{X}, \overline{X}] = \{\tilde{X} \in R^{n,n} \mid \underline{X} \leq \tilde{X} \leq \bar{X}\},$$

where $\underline{X}$, $\bar{X} \in R^{n,n}$, $\underline{X} \leq \bar{X}$.

IR^n is the set of all interval vectors

$$x := [\underline{x}, \bar{x}] := \{\tilde{x} \in R^n \mid \underline{x} \leq \tilde{x} \leq \bar{x}\},$$

where $\underline{x}, \bar{x} \in R^n$, $\underline{x} \leq \bar{x}$.

$\Box\Sigma := [\inf_{Z\in\Sigma} Z, \sup_{Z\in\Sigma} Z]$ for a bounded set Σ of $R^{n,n}$.

For all $X, Y \in IR^{n,n}$ and $x \in IR^n$ we have

$$X \pm Y = \Box\{\tilde{X} \pm \tilde{Y} \mid \tilde{X} \in X, \tilde{Y} \in Y\},$$

$$XY := \Box\{\tilde{X}\tilde{Y} \mid \tilde{X} \in X, \tilde{Y} \in Y\},$$

$$X^{-1} := \Box\{\tilde{X}^{-1} | \tilde{X} \in X\},$$

$$|X| := \sup\{|\tilde{X}| \; |\tilde{X} \in X\},$$

$$\langle x \rangle = \inf\{|\tilde{x}| \; |\tilde{x} \in x\},$$

$$\langle X \rangle := (x^*_{ij}), \text{ where } x^*_{ii} = \langle X_{ii} \rangle,$$

$$x^*_{ij} = -|X_{ij}|, i \neq j, \; X = (X_{ij}),$$

$$\rho(X) := \max\{|\lambda| \; |\lambda \text{ is an eigenvalue of } X\},$$

$$N := \{1, 2, \ldots n\},$$

$$N(i) := N \setminus \{i\},$$

$$N_1 := \{1, 2, \ldots k\},$$

$$N_2 := N \setminus N_1,$$

$$P_i(M) := \sum_{j\in N(i)} |m_{ij}|, i \in N,$$

$$P^*_i(M) := \sum_{j\in N(i)} |m_{ij}|, i \in N,$$

$$P_{i,\alpha}(M) := \alpha P_i(M) + (1-\alpha)P^*_i(M), \; i \in N \text{ and } \alpha \in [0,1],$$

$$\tilde{m}_i := P_{i,\alpha}(M),$$

$$C_1 := \{X \in IR^{n,n} : \langle X_{ii} \rangle > P_{i,\alpha}(X), \text{ with } \alpha \in [0,1], 0 \notin X_{ii}, i \in N\}.$$

Definition 1. *[4] A square matrix $X \in IR^{n,n}$ is an H - matrix if $\langle X \rangle u > 0$ for some positive vector $u \in R^n$.*

It is clear that $X \in C_1$ then X is a H - matrix and if $X \in C_1$ with $\alpha = 1$, X is a strictly diagonally dominant matrix.

We will need also the following results:

Lemma 1. *[4] If $X \in IR^{n,n}$ is an H - matrix, then $|X^{-1}| \leq \langle X^{-1} \rangle$.*

Lemma 2. *[4] Let $G, H \in IR^{n,n}$ satisfy $\rho(|G||H|) < 1$. Then for any $c \in IR^n$, the following statements hold:*

(i) The equation $x = G(Hx + c)$ has a unique solution $x \in IR^n$.

(ii) For any starting vector $x^{(0)} \in IR^n$, the iterative method

$$x^{(k+1)} = G(Hx^{(k)} + c), \;\; k = 0, 1, \ldots$$

converges to the solution x of $x = G(Hx + c)$.

(iii) If $x^{(1)} \subseteq x^{(0)}$, then for all $k \geq 1$, $x \subseteq x^{(k)} \subseteq x^{(k-1)} \subseteq \ldots \subseteq x^{(0)}$.

(iv) If $x^{(0)} \subseteq x^{(1)}$, then for all $k \geq 1$, $x^{(0)} \subseteq \ldots \subseteq x^{(k-1)} \subseteq x^{(k)} \subseteq x$.

2. Convergence Conditions

In this section we will establish some sufficient conditions of convergence for the interval MSOR method when the matrix $A \in IR^{n,n}$ belongs to the class C_1.

If we denote by

$$B^* = (1-w)D + (w-w')R + wT = \begin{bmatrix} (1-w)D_1 & -wT_1 \\ 0 & (1-w')D_2 \end{bmatrix}, \tag{2.1}$$

we have

$$M_{w,w'} = (D - w'S)^{-1}B^*. \tag{2.2}$$

. Let

$$M = \langle D - w'S \rangle^{-1}|B^*|, \tag{2.3.}$$

$$\tilde{s}_i = P_{i,\alpha}(S) \text{ and } \tilde{t}_i = P_{i,\alpha}(T).$$

If $A\langle_{ii}\rangle - |w'|\tilde{s}_i > 0, \; i \in N$, then it holds

$$\rho(M) \leq \max\{r_1, r_2\} \tag{2.4.}$$

where

$$r_1 = \max_{i \in N_1} \frac{|1-w||A_{ii}| + w|\tilde{t_i}}{\langle A_{ii}\rangle - |w'|\tilde{s}_i} \text{ and } r_2 = \max_{i \in N_2} \frac{|1-w||A_{ii}| + w|\tilde{t_i}}{\langle A_{ii}\rangle - |w'|\tilde{s}_i}.$$

Proof. Let λ be an eigenvalue of M, then we have

$$det(\lambda I - M) = 0,$$

which is equivalent to

$$(2.5.) \qquad detC = 0$$

with $C = \lambda\langle D\rangle - \lambda|w'||S| - |B^*|$.

Let us suppose that (2.4) is not valid, then we have

$$|\lambda|\left(\langle A_{ii}\rangle - |w'|\tilde{s}_i\right) > |1-w||A_{ii}| + |w|\tilde{t_i}, \;\; i \in N_1,$$

and

$$|\alpha|(\langle A_{ii}\rangle - |w'|\tilde{s}_i) > |1-w'||A_{ii}| + |w|\tilde{t_i}, \;\; i \in N_2,$$

or equivalently

$$|\lambda|\langle A_{ii}\rangle - |1-w||A_{ii}| > |w'||\lambda|\tilde{s}_i + |w|\tilde{t_i}, \;\; i \in N_1,$$

and

$$|\lambda|\langle A_{ii}\rangle - |1-w'|A_{ii} > |\lambda||w'|\tilde{s}_i + |w|\tilde{t_i}, \;\; i \in N_2.$$

These last relations imply

$$||\lambda|\langle A_{ii}\rangle - |1-w||A_{ii}|| > |w'||\lambda|\tilde{s}_i + |w|\tilde{t_i}, \;\; i \in N_1,$$

and

$$||\lambda|\langle A_{ii}\rangle - |1-w||A_{ii}|| > |\lambda||w'|\tilde{s}_i + |w|\tilde{t_i}, \;\; i \in N_2.$$

As

$$c_{ii} = \lambda\langle A_{ii}\rangle - |1-w||A_{ii}|, \;\in N_1,$$

and

$$c_{ii} = \lambda\langle A_{ii}\rangle - |1-w'||A_{ii}|, \;\; i \in N_2$$

we have

$$|c_{ii}| > P_{i,\alpha}(C), \in N, \text{ for some } \alpha \in [0,1],$$

So, by [2] C is a nonsingular matrix, which is a contradiction. □

Let us denote

$$\phi_i^0(w) = \frac{|A_{ii}| - \langle A_{ii}\rangle + w\tilde{t}_i}{|A_{ii}| - \tilde{s}_i}, \quad i \in N_2,$$

$$\phi_i^1(w) = \frac{\langle A_{ii}\rangle - |A_{ii}| + w(|A_{ii}| - \tilde{t}_i}{\tilde{s}_i}, \quad i \in N_1,$$

$$\phi_i^2(w) = \frac{|A_{ii}| - \langle A_{ii}\rangle - w\tilde{t}_i}{|A_{ii}| + \tilde{s}_i}, \quad i \in N_2,$$

$$\phi_i^3(w) = \frac{|A_{ii}| + \langle A_{ii}\rangle - w\left(|A_{ii}| + \tilde{t}_i\right)}{\tilde{s}_i}, \quad i \in N_1,$$

$$\Omega_i^0 = \frac{|A_{ii}| - \langle A_{ii}\rangle}{|A_{ii}| - \tilde{t}_i - \tilde{s}_i}, \quad i \in N,$$

$$\Omega_i^1 = \frac{|A_{ii}| + \langle A_{ii}\rangle - \tilde{s}_i}{|A_{ii}| + \tilde{t}_i}, \quad i \in N_1,$$

$$\Omega_i^2 = \frac{\langle A_{ii}\rangle - \tilde{s}_i}{\tilde{t}_i}, \quad i \in N_2.$$

In the following theorem we will use these functions to define the area of convergence of the interval MSOR method.

. If the matrix A belongs to the class C_1, then, for any starting vector $x^{(0)} \in R^n$, the interval MSOR method is convergent, i.e.,

$$x^{(k)} \to x^* \in IR^n(k \to \infty),$$

if

$$\Omega_0 < w < \min\{\Omega_1, \Omega_2\}, \tag{2.6}$$

and

$$g_0(w) < w' < \min\{g_1(w), g_2(w), g_3(w)\}, \tag{2.7}$$

where

$$\Omega_0 = \max_{i \in N} \Omega_i^0, \quad \Omega_1 = \min_{i \in N_1} \Omega_i^1,$$

$$\Omega_2 = min_{i\in N_2}\Omega_i^2, \quad g_0(w) = \max_{i \in N_2} \phi_i^0(w),$$

$$g_1(w) = \min_{i\in N_1} \phi_i^1(w), \qquad g_2(w) = \min_{i\in N_2} \phi_i^2(w),$$

$$g_3(w) = \min_{i\in N_1} \phi_i^3(w).$$

Furthermore, $x \supseteq \{\tilde{x} \mid \tilde{A}\tilde{x} = \tilde{b}, \tilde{A} \in A, \tilde{b} \in b, \tilde{A} \in R^{n,n}, \tilde{b}, \tilde{x} \in R^n\}$.

Proof. From Lemma 2 the interval MSOR method is convergent if $\rho < (|M_{w,w'}|) < 1$. From [4], we have

$$|M_{w,w'}| \leq \langle D - w'S\rangle^{-1}|(1-w)D + (w-w')R + wT|.$$

Then $\rho(|M_{w,w'}|) \leq \rho(M)$, with M given by (2.3).

From theorem 1, $\mu(M) < 1$ if $\max\{r_1, r_2\} < 1$ and therefore the interval MSOR method is convergent.

Thus, we will verify that, with w and w' given by (2.6) and (2.7), for all $i \in N$, we have

$$\langle A_{ii}\rangle - w'\tilde{s}_i > 0 \tag{2.8}$$

and

$$\max\{r_1, r_2\} < 1. \tag{2.9}$$

In order to prove (2.8), for all $i \in N$, from (2.6) and (2.7) we have:

(i) For $i \in N_2$, if $w' \leq 1$, from $g_0(w) < w'$ we get (2.8) and if $w' \geq 1$ the same inequality can be obtained from $w' < g_2(w)$.

(ii) For $i \in N_1$, if $w \leq 1$, from $w' < g_1(w)$ we obtain (2.8) and if $w \geq 1$ the same inequality can be got from $w' < g_3(w)$.

To obtain values of w and w' for which (2.9) is verified we have to consider several cases:

(I) if $w \in (0,1]$ and $w'(0,1]$ from (2.7) we have

$$w' < \phi_i^1(w), i \in N_1 \text{ and } w' > \phi_i^0(w), i \in N_2, \tag{2.10}$$

which is equivalent to (2.9).

(II) If $w \in (0,1]$ and $w' \geq 1$, from (2.7) we have

$$w' < \phi_i^1(w), i \in N_1 \textit{ and } w' < \phi_i^2(w),\ i \in N_2, \tag{2.11}$$

therefore we get (2.9).

(III) If $w \geq 1$ and $w' \in [0,1]$, from (2.7) we have

$$w' < \phi_i^3(w), i \in N_1 \text{ and } w' > \phi_i^0(w), i \in N_2, \tag{2.12}$$

which is equivalent to (2.9).

(IV) If $w \geq 1$ and $w' \geq 1$, from (2.7) we have

$$w' < \phi_i^3(w), i \in N_1 \text{ and } w' < \phi_i^2(w), i \in N_2. \tag{2.13.}$$

In order to verify that we can always find value for w and w' in (2.6) and (2.7), we can firstly see that:

$$g_1(w) \leq g_3(w), \text{ if } w \leq 1$$

and

$$g_3(w) \leq g_1(w), \text{ if } w \geq 1.$$

We also have

$$\Omega_1^0 < 1 < \Omega_i^1, \ i \in N_1, \ \Omega_i^0 < 1 < \Omega_i^2, i \in N_2,$$

since A belongs to the class C_1 or which is equivalent

$$\langle A_{ii} \rangle - \tilde{t}_i - \tilde{s}_i > 0, \ \ i \in N.$$

Let us consider now $w \in (\Omega_i^0, 1], i \in N$. Now we have:

$$\phi_i^0(w) < w, \ \ i \in N_2, w < \phi_i^1(w), \ \ i \in N_1 \text{ and } 1 < \phi_i^2(w), \ i \in N_2.$$

Then it follows that

$$g_0(w) < \min\{g_1(w), g_2(w)\}, \text{ if } w \in (\Omega_0, 1].$$

Therefore, for $w \in (\Omega_0, 1]$, we can always find values for w' so that the interval MSOR method is convergent.

If $w \in [1, \min(\Omega_1, \Omega_2\})$ then it holds

$$w \in [1, \Omega_i^2), \quad \phi_i^0(w) < 1 < \phi_i^2(w), \ \ i \in N_2,$$

and

$$w \in [1, \Omega_i^1) \text{ and } \phi_i^3(w) > 1 i \in N_1.$$

As above we also conclude that for $w \in [1, \min(\Omega_1, \Omega_2)$ we can always find values for w' so that the interval MSOR method is convergent.

Thus, by Lemma 2, the interval MSOR method converges to $x^* \in IR^n$ for an starting vector $x^{(0)} \in IR^n$, i.e., $x^* = M_{w,w'}x^* + d_{w,w'}$.

In order to prove

$$x^* \supseteq \{\tilde{x} \mid \tilde{A}\tilde{x} = \tilde{b},\ \tilde{A} \in A,\ \tilde{b} \in b,\ \ \tilde{A} \in R^{n,n}, \tilde{b}, \tilde{x} \in R^n\}.$$

following [1], let us suppose that $\tilde{A}\tilde{x} = \tilde{b}$, with $\tilde{A} \in A$ and $\tilde{b} \in b$.

Therefore we have

$$\tilde{A} = \tilde{D} + \tilde{T} + \tilde{S}, \text{ where } \tilde{D} \in D, \tilde{T} \in T \text{ and } \tilde{S} \in S.$$

Thus

$$\tilde{x} = \tilde{M}_{w,w'}\tilde{x} + \tilde{d}_{w,w'}\tilde{x} + d_{w,w'}.$$

Choosing $x^{(0)} = \tilde{x}$ we have $\tilde{x} = x^{(0)} \in x^{(1)}$ and from Lemma 2 we obtain the desired result.□

If we make $\alpha = 1$ we have the following

Corollary 1. *If A is a strictly diagonally dominant (SDD) matrix then for any starting vector $x^{(0)} \in IR^n$, the interval MSOR method is convergent if*

$$\max_{i \in N_2} \frac{|A_{ii}| - \langle A_{ii} \rangle}{|A_{ii}| - S_i} < w' < \min_{i \in N_2} \frac{|A_{ii}| + \langle A_{ii} \rangle}{|A_{ii}| + S_i}$$

and

$$\max_{i \in N} \frac{|A_{ii}| - \langle A_{ii} \rangle}{|A_{ii}| - t_i - s_i} < w < \min_{i \in N_1} \frac{|A_{ii}| + \langle A_{ii} \rangle}{\langle A_{ii} \rangle + t_i},$$

where t_i and s_i are the sums of the absolute values of the entries of the ith row of the matrices T and S, respectively.

Following [1] we can extend our results for H- matrices.

. Let $A \in IR^{n,n}$ be an H - matrix and $\Lambda \in R^{n,n}$ be a positive diagonal matrix for which $A\Lambda$ is an SDD matrix. Then the interval MSOR method converges to $x^* \in IR^n$ for any starting $x^{(0)} \in IR^n$, provided we choose w, w' an in theorem 1, taking elements of the matrix $A\Lambda$ instead of elements of the matrix A. Furthermore, $x^* \supseteq \{\tilde{x} \mid \tilde{A}\tilde{x} = \tilde{b},\ \ \tilde{A} \in A,\ \ \tilde{b} \in b,\ \ \tilde{A} \in R^{n,n},\ \tilde{b} \in R^n\}$.

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