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# A THEOREM ON COINCIDENCE POINT FOR A FAMILY OF MAPPINGS

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**Abstract**

In [8] Lj.Ćirić proved the following theorem on coincidence point:

*Let* $(X,d)$ *be a metric space,* $T : X \to CL(X)$ *(the family of nonempty, closed subsets of* $X$*) and* $I : X \to X$ *such that* $T(X) \subseteq I(X)$. *If* $I(X)$ *is* $(T,I)$ *orbitally complete and for each* $x,y \in X$

$$H(Tx,Ty) \le a \max\Big\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), \frac{1}{2}\Big[d(Ix,Ty)+ d(Iy,Tx)\Big]\Big\}+b\min\Big\{\max\{d(Ix,Tx), d(Iy,Ty)\}, \frac{1}{2}\Big[d(Ix,Ty)+d(Iy,Tx)\Big]\Big\}$$

*where* $a,b \in \mathbf{R}$, $a \ge 0$, $b > 0$, $a+b=1$, *then there exists* $z \in X$ *such that* $Iz \in Tz$.

In this paper we shall give a generalization of this theorem for a family of multivalued mappings $\{T_n\}_{n\in\mathbf{N}}$.

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## 1. Introduction

J.Markin [13] and S.B. Nadler [14] extended Banach fixed point theorem for multivalued mappings. Since 1968 many fixed point theorems for multivalued mappings $T : X \to CL(X)$ are proved, where $T$ satisfies a more general condition then

$$H(Tx,Ty) \leq ad(x,y),\ x,y \in X,\ a \in [0,1).$$

In this paper a coincidence point theorem for a family of multivalued mappings will be proved, which generalizes fixed point and coincidence point theorems of [1] - [9], [11] - [20].

## 2. Main result

**Definition 2.1.** *Let $(X,d)$ be a metric space, $T_n : X \to \mathcal{P}(X)$ $(n \in \mathbf{N})$ and $I : X \to X$. If for a point $x_0 \in X$ there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ in $X$ such that $Ix_{n+1} \in T_{n+1}x_n$, $n \in \mathbf{N} \cup \{0\}$ then $\{Ix_n;\ n \in \mathbf{N}\}$ is an orbit for $(\{T_n\}_{n\in\mathbf{N}}, I)$ at $x_0$.*

**Definition 2.2.** *Let $(X,d)$ be a metric space, $T_n : X \to \mathcal{P}(X)$ $(n \in \mathbf{N})$ and $I : X \to X$. $X$ is $(\{T_n\}_{n\in\mathbf{N}}, I)$ orbitally complete if every Cauchy sequence $\{Ix_n\}_{n\in\mathbf{N}}$, where $\{Ix_n;\ n \in \mathbf{N}\}$ is an orbit for $(\{T_n\}_{n\in\mathbf{N}}, I)$ at arbitrary $x_0 \in X$, converges in $X$.*

**Theorem 2.1.** *Let $(X,d)$ be a metric space, $T_n : X \to CL(X)$ $(n \in \mathbf{N})$ and $I : X \to X$ such that $T_n(X) \subseteq I(X)$, for every $n \in \mathbf{N}$ and for every $(i,j) \in \mathbf{N} \times \mathbf{N} \setminus \Delta$*

$$(1) \qquad H(T_ix,T_jy) \leq a \max\Big\{d(Ix,Iy), d(Ix,T_ix), d(Iy,T_jy), \frac{1}{2}\Big[d(Ix,T_jy) + d(Iy,T_ix)\Big]\Big\}$$
$$+b \min\{\max\{d(Ix,T_ix), d(Iy,T_jy)\}, \frac{1}{2}[d(Ix,T_jy) + d(Iy,T_ix)\},$$

*for every $x,y \in X$ where $a,b \in \mathbf{R}$, $a \geq 0$, $b > 0$, $a+b = 1$.*

*Suppose that one of the following two conditions is satisfied:*

*(i) $I(X)$ is $(\{T_n\}_{n\in\mathbf{N}}, I)$ orbitally complete.*

*(ii) $I$ is continuous and $IT_nx \subseteq T_nIx$, for every $n \in \mathbf{N}$ and every $x \in X$.*

*Then there exists $z \in X$ such that*

$$Iz \in \bigcap_{n\in\mathbf{N}} T_nz.$$

*Proof.* Let $x_0 \in X$ and $x_1 \in X$ such that $Ix_1 \in T_1x_0$ and let $y_1 = Ix_1$. If $c = (1+b^2/2)^{1/2}$ then $c > 1$ and so there exists $x_2 \in X$ such that $Ix_2 \in T_2x_1$ and

$$d(y_1, Ix_2) \leq cd(y_1, T_2x_1).$$

Let $y_2 = Ix_2$. Continuing this process we obtain two sequences $\{x_n\}_{n\in \mathrm{N}}$ and $\{y_n\}_{n\in \mathrm{N}}$ such that $y_{n+1} = Ix_{n+1} \in T_{n+1}x_n$ $(n \in \mathrm{N} \cup \{0\})$ and

(2). $$d(y_n, y_{n+1}) \leq cd(y_n, T_{n+1}x_n) \; (n \in \mathrm{N})$$

Further, we have:

$$d(y_{n+1}, T_{n+2}x_{n+1}) \leq H(T_{n+1}x_n, T_{n+2}x_{n+1}) \leq$$

$$\leq a \max\{d(Ix_n, Ix_{n+1}), M(x_n, x_{n+1}),$$

$$N(x_n, x_{n+1})\} + b \min\{M(x_n, x_{n+1}),$$

$$N(x_n, x_{n+1})\}$$

where:

$$M(x_n, x_{n+1}) = \max\{d(Ix_n, T_{n+1}x_n), d(Ix_{n+1}, T_{n+2}x_{n+1})\}$$

$$N(x_n, x_{n+1}) = \frac{1}{2}[d(Ix_n, T_{n+2}x_{n+1}) + d(Ix_{n+1}, T_{n+1}x_n)]$$

$$= \frac{1}{2}d(Ix_n, T_{n+2}x_{n+1}).$$

Hence

$$M(x_n, x_{n+1}) \leq \max\{d(y_n, y_{n+1}), d(y_{n+1}, T_{n+2}x_{n+1})\}$$

$$N(x_n, x_{n+1}) = \frac{1}{2}d(y_n, T_{n+2}x_{n+1}) \leq$$

$$\leq \frac{1}{2}[d(y_n, y_{n+1}) + d(y_{n+1}, T_{n+2}x_{n+1})]$$

which implies that

(3) $$d(y_{n+1}, T_{n+2}x_{n+1}) \leq a \max\{d(y_n, y_{n+1}),$$

$$d(y_{n+1}, T_{n+2}x_{n+1})\} + \frac{b}{2}\Big[d(y_n, y_{n+1}) + d(y_{n+1}, T_{n+2}x_{n+1})\Big]$$

We shall prove that for every $n \in \mathbf{N}$

$$d(y_{n+1}, T_{n+2}x_{n+1}) \leq d(y_n, y_{n+1}). \tag{4}$$

If (4) does not hold for some $n \in \mathbf{N}$ i.e.

$$d(y_{n+1}, T_{n+2}x_{n+1}) > d(y_n, y_{n+1})$$

we have from (3) that

$$d(y_{n+1}, T_{n+2}x_{n+1}) < (a+b)d(y_{n+1}, T_{n+2}x_{n+1})$$

which is a contradiction.

So, from (2) we obtain that

$$d(y_{n+1}, y_{n+2}) \leq cd(y_n, y_{n+1}), \tag{5}$$

and using the triangle inequality we conclude that

$$d(y_n, y_{n+2}) \leq 2cd(y_n, y_{n+1}). \tag{6}$$

From (4), (5) and (6) it follows easily, using

$$\max\{d(y_n, T_{n+1}x_n), d(y_{n+2}, T_{n+3}x_{n+2})\} \leq$$

$$\leq \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} \leq cd(y_n, y_{n+1}),$$

$$\frac{1}{2}[d(y_n, T_{n+3}x_{n+2}) + d(y_{n+2}, T_{n+1}x_n)] \leq$$

$$\leq \frac{1}{2}[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, T_{n+3}x_{n+2}) +$$

$$+ d(y_{n+2}, y_{n+1})] \leq \frac{1}{2}(1+3c)d(y_n, y_{n+1}) \leq 2cd(y_n, y_{n+1})$$

that

$$H(T_{n+1}x_n, T_{n+3}x_{n+2}) \leq c(2a+b)d(y_n, y_{n+1}) = c(2-b)d(y_n, y_{n+1}). \tag{7}$$

By (1), (4), (5) and (7) we obtain that

$$H(T_{n+2}x_{n+1}, T_{n+3}x_{n+2}) \leq$$

$$\leq a \cdot cd(y_n, y_{n+1}) + b(1-b/2)cd(y_n, y_{n+1}) =$$

$$= (a + b - b^2/2)cd(y_n, y_{n+1})$$
$$= (1 - b^2/2)cd(y_n, y_{n+1})$$

and so

$$d(y_{n+2}, y_{n+3}) \leq cd(y_{n+2}, T_{n+3}x_{n+2})$$
$$\leq cH(T_{n+2}x_{n+1}, T_{n+3}x_{n+2})$$
$$\leq (1 - b^2/2)c^2 d(y_n, y_{n+1}).$$

Hence

$$d(y_{n+2}, y_{n+3}) \leq (1 - b^4/4)d(y_n, y_{n+1}) \tag{8}$$

and (8) implies

$$d(y_n, y_{n+1}) \leq (1 - b^4/4)^{[n/2]}(1 + b^2/2)^{1/2} d(y_0, y_1). \tag{9}$$

From (9) we conclude that $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence.

Let (i) be satisfied.

Since $I(X)$ is $(\{T_n\}_{n\in\mathbf{N}}, I)$ orbitally complete and $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence of the form

$$y_{n+1} = Ix_{n+1} \in T_{n+1}x_n \quad (n \in \mathbf{N} \cup \{0\})$$

it follows that there exists $p \in IX$ such that $p = \lim_{n\to\infty} y_n = Iz$, for some $z \in X$. We shall prove that $Iz \in \bigcap_{n\in\mathbf{N}} T_n z$ i.e. $d(Iz, T_n z) = 0$, for every $n \in \mathbf{N}$. Suppose that $d(Iz, T_n z) > 0$.

Let $n \in \mathbf{N}$ and $m \in \mathbf{N}$ such that $m + 1 \neq n$. Then from (1) we have

$$d(Iz, T_n z) = d(p, T_n z) \leq d(p, y_{m+1}) +$$

$$+ d(y_{m+1}, T_n z) \leq d(p, y_{m+1}) + H(T_{m+1}x_m, T_n z) \leq$$
$$\leq d(p, y_{m+1}) + a \max\{d(y_m, p), d(y_m, T_{m+1}x_m),$$
$$d(Iz, T_n z), \frac{1}{2}[d(y_m, T_n z) + d(p, T_{m+1}x_m)]\}$$
$$+ b \min\{\max\{d(y_m, T_{m+1}x_m), d(Iz, T_n z)\},$$
$$\frac{1}{2}[d(y_m, T_n z) + d(p, T_{m+1}x_m)]\} \leq$$

$$\leq d(p,y_{m+1})+a\max\{d(y_m,p),d(y_m,y_{m+1}),d(Iz,T_nz),$$
$$\frac{1}{2}[d(y_m,T_nz)+d(p,y_{m+1})]\}$$
$$+b\min\{\max\{d(y_m,y_{m+1}),d(Iz,T_nz)\},$$
$$\frac{1}{2}[d(y_m,T_nz)+d(p,y_{m+1})]\}.$$

Since $\lim\limits_{m\to\infty} d(y_m,y_{m+1})=0$, $\lim\limits_{m\to\infty} d(p,y_{m+1})=0$ and $\lim\limits_{m\to\infty} d(y_m,T_nz)=d(Iz,T_nz)$, for sufficiently large $m\in\mathbf{N}$ we have that

$$\max\{d(y_m,p),d(y_m,y_{m+1}),d(Iz,T_nz),$$
$$\frac{1}{2}[d(y_m,T_nz)+d(p,y_{m+1})]\}=d(Iz,T_nz)$$

and

$$\min\{\max\{d(y_m,y_{m+1}),d(Iz,T_nz)\},\frac{1}{2}[d(y_m,T_nz)+d(p,y_{m+1}]\}=$$
$$=\frac{1}{2}[d(y_m,T_nz)+d(p,y_{m+1})].$$

Hence, when $m\to\infty$ we obtain that

$$d(Iz,T_nz)\leq ad(Iz,T_nz)+\frac{b}{2}d(Iz,T_nz)=$$
$$=(a+\frac{b}{2})d(Iz,T_nz)<d(Iz,T_nz)$$

which is a contradiction. So, $d(Iz,T_nz)=0$ and since $T_nz\in CL(X)$ it follows that

$$Iz\in\bigcap_{n\in\mathbf{N}}T_nz.$$

Let (ii) be satisfied and $d(Iz,T_mz)>0$. Since $Ix_n\in T_nx_{n-1}$ $(n\in\mathbf{N})$ and $IIx_n\in IT_nx_{n-1}\subseteq T_nIx_{n-1}$ $(n\in\mathbf{N})$ we have from (1) that $(m\neq n):$

$$d(IIx_n,T_mz)\leq H(T_nIx_{n-1},T_mz)\leq$$
$$\leq a\max\{d(IIx_{n-1},Iz),d(IIx_{n-1},T_nIx_{n-1}),d(Iz,T_mz),$$
$$\frac{1}{2}[d(IIx_{n-1},T_mz)+d(Iz,T_nIx_{n-1})]\}$$

$$+b\min\{\max\{d(IIx_{n-1},T_nIx_{n-1}),d(Iz,T_mz)\},$$
$$\frac{1}{2}[d(IIx_{n-1},T_mz)+d(Iz,T_nIx_{n-1})]\}$$

Since

$$\lim_{n\to\infty} d(IIx_{n-1},Iz)=0,\ \lim_{n\to\infty} d(IIx_{n-1},IIx_n)=0$$
$$\lim_{n\to\infty} d(IIx_{n-1},T_mz)=d(Iz,T_mz)$$

for sufficiently large $n\in\mathbf{N}$ we have that

$$\max\{d(IIx_{n-1},Iz),d(IIx_{n-1},IIx_n),$$
$$d(Iz,T_mz),\frac{1}{2}[d(IIx_{n-1},T_mz)+d(Iz,IIx_n)]\}=d(Iz,T_mz),$$
$$\min\{\max\{d(IIx_{n-1},IIx_n),d(Iz,T_mz)\},\frac{1}{2}[d(IIx_{n-1},T_mz)+d(Iz,IIx_n)]\}=$$
$$=\frac{1}{2}d(Iz,T_mz).$$

Hence, for sufficiently large $n\in\mathbf{N}$

$$d(IIx_n,T_mz)\leq(a+b/2)d(Iz,T_mz).$$

When $n\to\infty$ we obtain

$$d(Iz,T_mz)\leq(a+b/2)d(Iz,T_mz)<d(Iz,T_nz)$$

which is a contradiction. Hence

$$Iz\in\bigcap_{n\in\mathbf{N}}T_nz.$$

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