NOVI SAD J. MATH.
VOL. 26, NO. 1, 1996, 11-17

RELATED FIXED POINT THEOREMS FOR THREE METRIC SPACES

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Abstract

We obtained some related fixed point theorems for three metric spaces which generalize some results of [1] from two metric spaces to three metric spaces

AMS Mathematics Subject Classification (1991): 54H25
Key words and phrases: three metric spaces, fixed points

The following fixed point theorem was proved by Fisher ([1]).

Theorem 1. *Let (X,d) and (Z,σ) be complete metric spaces. If S is a continuous mapping of X into Z, and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} d(RSx, RSx') &\leq c\max\{d(x,x'), d(x,RSx), d(x',RSx'), \sigma(Sx,Sx')\}, \\ \sigma(SRz, SRz') &\leq c\max\{\sigma(z,z'), \sigma(z,SRz), \sigma(z',SRz'), d(Rz,Rz')\} \end{aligned}$$

for all x, x' in X and z, z' in Z, where $0 \leq c < 1$, then RS has a unique fixed point u in X and SR has a unique fixed point w in Z. Further, $Su = w$ and $Rw = u$.

We now prove the following related fixed point theorem which generalizes Theorem 1:

Theorem 2. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned}
(1)\quad d(RSTx, RSTx') &\leq c\max\{d(x,x'), d(x,RSTx), d(x',RSTx'),\\
&\qquad \rho(Tx,Tx'), \sigma(STx,STx')\},\\
(2)\quad \rho(TRSy, TRSy') &\leq c\max\{\rho(y,y'), \rho(y,TRSy), \rho(y',TRSy'),\\
&\qquad \sigma(Sy,Sy'), d(RSy,RSy')\},\\
(3)\quad \sigma(STRz, STRz') &\leq c\max\{\sigma(z,z'), \sigma(z,STRz), \sigma(z',STRz'),\\
&\qquad d(Rz,Rz'), \rho(TRz,TRz')\},
\end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z where $0 \leq c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, and $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively by

$$x_n = (RST)^n x_0, \quad y_n = Tx_{n-1}, \quad z_n = Sy_n$$

for $n = 1, 2, \ldots$.

Applying inequality (2) we have

$$\begin{aligned}
(4)\,\rho(y_n, y_{n+1}) &= \rho(TRSy_{n-1}, TRSy_n)\\
&\leq c\max\{\rho(y_{n-1},y_n), \rho(y_n,y_{n+1}), \sigma(z_{n-1},z_n), d(x_{n-1},x_n)\}\\
&= c\max\{d(x_{n-1},x_n), \rho(y_{n-1},y_n), \sigma(z_{n-1},z_n)\}.
\end{aligned}$$

Using inequality (3) we have

$$\begin{aligned}
(5)\,\sigma(z_n, z_{n+1}) &= \sigma(STRz_{n-1}, STRz_n)\\
&\leq c\max\{\sigma(z_{n-1},z_n), \sigma(z_n,z_{n+1}), d(x_{n-1},x_n), \rho(y_n,y_{n+1})\}\\
&= c\max\{d(x_{n-1},x_n), \rho(y_{n-1},y_n), \sigma(z_{n-1},z_n)\}
\end{aligned}$$

on using inequality (4).

Using inequality (1) we have

$$(6)\, d(x_n, x_{n+1}) = d(RSTx_{n-1}, RSTx_n)$$
$$\leq c\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})\}$$
$$= c\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n)\},$$

on using inequalities (4) and (5).

It now follows easily by induction on using inequalities (4), (5) and (6) that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}, \\ \rho(y_n, y_{n+1}) &\leq c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}, \\ \sigma(z_n, z_{n+1}) &\leq c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}. \end{aligned}$$

Since $c < 1$, it follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits u, v and w in X, Y and Z respectively. Since T and S are continuous, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tu = v, \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} Sy_n = Sv = w.$$

Using inequality (1) again we have

$$\begin{aligned} d(RSTu, x_n) &= d(RSTu, RSTx_{n-1}) \\ &\leq c\max\{d(u, x_{n-1}), d(u, RSTu), d(x_{n-1}, x_n), \\ &\qquad \rho(Tu, Tx_{n-1}), \sigma(STu, STx_{n-1})\}. \end{aligned}$$

Since T and S are continuous, it follows on letting n tend to infinity that

$$d(RSTu, u) \leq cd(u, RSTu)$$

Thus $RSTu = u$, since $c < 1$ and so u is a fixed point of RST.

We now have

$$TRSv = TRSTu = Tu = v$$

and so

$$STRw = STRSv = Sv = w.$$

Hence v and w are fixed points of TRS and STR respectively.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'. Then using inequality (1), we have

$$\begin{aligned} d(u,u') &= d(RSTu, RSTu') \\ &\leq c\max\{d(u,u'), d(u,RSTu), d(u',RSTu'), \rho(Tu,Tu'), \sigma(STu,STu')\} \\ &= c\max\{\rho(Tu,Tu'), \sigma(STu,STu')\}. \end{aligned}$$

Further, using inequality (2), we have

$$\begin{aligned} \rho(Tu,Tu') &= \rho(TRSTu, TRSTu') \\ &\leq c\max\{\rho(Tu',Tu), \rho(Tu,TRSTu), \rho(Tu',TRSTu'), \\ &\qquad \sigma(STu,STu'), d(RSTu,RSTu')\} \\ &= c\max\{d(u,u'), \sigma(STu,STu')\}. \end{aligned}$$

Hence we have

$$d(u,u') \leq c\sigma(STu,STu')$$

and finally, on using inequality (3), we now have

$$\begin{aligned} d(u,u') &\leq c\sigma(STu,STu') = c\sigma(STRSTu, STRSTu') \\ &\leq c^2\max\{\sigma(STu,STu'), \sigma(STu,STRSTu), \sigma(STu',STRSTu'), \\ &\qquad d(RSTu,RSTu'), \rho(TRSTu,TRSTu')\} \\ &= c^2\max\{\sigma(STu,STu'), d(u,u'), \rho(Tu,Tu')\} \\ &= c^2 d(u,u'). \end{aligned}$$

Since $c < 1$, it follows that $u = u'$ and the uniqueness of u follows.

Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR.

We finally prove that we also have $Rw = u$. To do this, note that

$$Rw = R(STRw) = RST(Rw)$$

and so Rw is a fixed point of RST. Since u is the unique fixed point of RST, it follows that $Rw = u$. This completes the proof of the theorem. □

Note that if we let $(X,d) = (Y,\rho)$ and let T be the identity mapping on X, we see that Theorem 2 reduces to Theorem 1.

To show that two of the mappings in Theorem 2 need to be continuous, let $X = Y = Z$ be the closed interval $[0, 1]$ with the usual metric, let $Tx = x$ for all x in $[0, 1]$ and let

$$Rx = Sx = \begin{cases} 1, & x = 0, \\ \frac{1}{2}x, & x \neq 0 \end{cases}.$$

Then T is continuous but S and R are not continuous. Further, it is easily seen that

$$RSTx = TRSx = STRx = \begin{cases} \frac{1}{2}, & x = 0, \\ \frac{1}{4}x, & x \neq 0 \end{cases}$$

and so inequalities (1), (2) and (3) are satisfied with $c = \frac{1}{2}$ but RST, TRS, and STR have no fixed points.

We finally prove an analogous result for compact metric spaces.

Theorem 3. *Let (X, d), (Y, ρ) and (Z, σ) be compact metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X satisfying the inequalities*

$$\begin{aligned}
(7)\, d(RSTx, RSTx') &< \max\{d(x,x'), d(x, RSTx), d(x', RSTx'), \\
&\qquad \rho(Tx, Tx'), \sigma(STx, STx')\}, \\
(8)\, \rho(TRSy, TRSy') &< \max\{\rho(y,y'), \rho(y, TRSy), \rho(y', TRSy'), \\
&\qquad \sigma(Sy, Sy'), d(RSy, RSy')\}, \\
(9)\, \sigma(STRz, STRz') &< \max\{\sigma(z,z'), \sigma(z, STRz), \sigma(z', STRz'), \\
&\qquad d(Rz, Rz'), \rho(TRz, TRz')\},
\end{aligned}$$

for all distinct x, x' in X, all distinct y, y' in Y, and all distinct z, z' in Z. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

Proof. Suppose first of all that there exists u, u' in X such that

$$\max\{d(u,u'), d(u, RSTu), d(u', RSTu'), \rho(Tu, Tu'), \sigma(STu, STu')\} = 0.$$

Then it follows immediately that $u = u'$ and $RSTu = u$ and on putting $Tu = v$, $Sv = w$, we have

$$RSv = u \Rightarrow TRSv = Tu = v,$$
$$STRSv = STRw = Sv = w \Rightarrow RSv = Rw = u.$$

The results of the theorem therefore hold in this case.

Similarly, if there exist v, v' in Y such that

$$\max\{\rho(v,v'), \rho(v,TRSv), \rho(v',TRSv'), \sigma(Sv,Sv'), d(RSv,RSv')\} = 0$$

or if there exist w, w' in Z such that

$$\max\{\sigma(w,w'), \sigma(w,STRw), \sigma(w',STRw'), d(Rw,Rw'), \rho(TRw,TRw')\} = 0,$$

then the results of the theorem again hold.

Now suppose that it is possible for no such u, u' or v, v' or w, w' to exist. Then

$$\max\{d(x,x'), d(x,RSTx), d(x',RSTx'), \rho(Tx,Tx'), \sigma(STx,STx')\} \neq 0$$

for all x, x' in X and so the real valued function $f(x,x')$ defined on $X \times X$ by

$$f(x,x') = \frac{d(RSTx,RSTx')}{\max\{d(x,x'), d(x,RSTx), d(x',RSTx'), \rho(Tx,Tx'), \sigma(STx,STx')\}}$$

is continuous. Since $X \times X$ is compact, f attains its maximum value c_1. Because of inequality (7), $c_1 < 1$ and so

$$\begin{aligned} d(RSTx,RSTx') \leq\ & c_1 \max\{d(x,x'), d(x,RSTx), d(x',RSTx'), \\ & \rho(Tx,Tx'), \sigma(STx,STx')\}, \end{aligned}$$

for all x, x' in X.

Similarly, there exist $c_2, c_3 < 1$ such that

$$\begin{aligned} \rho(TRSy,TRSy') \leq\ & c_2 \max\{\rho(y,y'), \rho(y,TRSy), \rho(y',TRSy'), \\ & \sigma(Sy,Sy'), d(RSy,RSy')\}, \end{aligned}$$

for all y, y' in Y and

$$\begin{aligned} \sigma(STRz,STRz') \leq\ & c_3 \max\{\sigma(z,z'), \sigma(z,STRz), \sigma(z',STRz'), \\ & d(Rz,Rz'), \rho(TRz,TRz')\}, \end{aligned}$$

for all z, z' in Z. It follows that the conditions of Theorem 2 are satisfied with $c = \max\{c_1, c_2, c_3\}$ and so the results of the theorem are again satisfied.

The uniqueness of u, v and w follows easily. □

Note again that if we let $(X,d) = (Y,\rho)$ and let T be the identity mapping on X in Theorem 3, it reduces to Theorem 4 of [1].

References

[1] Fisher, B., Related fixed points on two metric spaces, Math. Sem. Notes, Kobe Univ., **10** (1982), 17–26.

Received by the editors January 12, 1995.