Novi Sad J. Math.
Vol. 26, No. 1, 1996, 25–51

REMARKS ON NIKODÝM'S BOUNDEDNESS THEOREM

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Abstract

We extend the class of measures defined on orthomodular lattices or posets, orthoalgebras, and on effect algebras for which the Nikodým boundedness theorem holds. We present variants for finitely additive measures and completely additive measures. As a corollary, we obtain Nikodým's boundedness theorem for measures defined on the system $L(H)$ of all closed subspaces of a real or complex Hilbert space H. This result will be proved without using the Bair category theorem.

AMS Mathematics Subject Classification (1991): 46G12, 81P10
Key words and phrases: Nikodým boundedness theorem, orthoalgebra, orthomodular poset, orthomodular lattice, measure, regular measure, Hilbert space, inner product space

1. Introduction

The Nikodým boundedness theorem is one of the most important results of measure theory, and it says [7] that if a system $\mathcal{M}$ of σ-additive measures

[1]This research is supported by the grant G-229/94 of the Slovak Academy of Sciences, Slovakia.

on a σ-algebra $\mathcal{S}$ of subsets of a non-void set Ω, is pointwisely bounded, i.e., for any $E \in \mathcal{S}$, there is a positive constant $K(E)$ such that

$$|\mu(E)| \leq K(E) \quad \text{for all } \mu \in \mathcal{M},$$

then $\mathcal{M}$ is uniformly bounded, i.e., there is a positive constant K such that

$$|\mu(E)| \leq K \quad \text{for all } \mu \in \mathcal{M} \text{ and all } E \in \mathcal{S}.$$

The original proof of Nikodým is based on the Bair category theorem [7], Thm IV.9.8 and on the fact that any σ-algebra $\mathcal{S}$ is distributive. It is known that the Bair category theorem implies the principle of dependent choice [3] in set theory, and it is related to a weaker form of axiom of choice. So, any "elementary" proof (i.e. without using the Bair category theorem) is very desirable. Such a successful approach is an application of the Mikusiński–Antosik diagonal theorem [15], [18].

Any attempt to extend the Nikodým boundedness theorem for quantum structures (structures like orthomodular posets, orthomodular lattices, orthoalgebras or effect algebras) strikes against the distributivity which fails, as usually, in these structures. Therefore, we have finite σ-additive measures which are unbounded, so Nikodým's boundedness theorem can fail even for a system consisting of one σ-additive measure. Such a situation is, for example, in the most important case of quantum structures, the system $L(H)$ of all closed subspaces of a real or complex Hilbert space H. Gleason [11] (see also [8], Prop. 3.2.4) showed that for any finite-dimensional Hilbert space H, $\dim H \geq 2$, there are plenty of unbounded σ-additive measures on $L(H)$.

Anyway, there are some generalizations of Nikodým's boundedness theorem for different types of quantum structures: de Lucia and Pap [16] for D-posets (effect algebras), de Maria and Morales [17], Hamhalter [13] for the projection lattice of a von Neumann algebra, the author for $L(H)$ [9].

In the present article we generalize the Nikodým boundedness theorem for measures on orthomodular lattices having a Gleason property and for regular measures. As corollaries we obtain new proofs for Gleason measures on $L(H)$ and on the projection lattice of a von Neumann algebra (see [13], [9]).

2. Measures on effect algebras

An *effect algebra* is a non-empty set L with two particular elements 0, 1, and with a partial binary operation $\oplus : L \times L \to L$ such that for all $a, b, c \in L$ we have

(EAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity);

(EAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);

(EAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);

(EAiv) if $1 \oplus a$ is defined, then $a = 0$ (zero-one law).

Let a and b be two elements of an effect algebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^\perp$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a and b. It is evident that

$$b \ominus a = (a \oplus b^\perp)^\perp.$$

An *atom* of L is a non-zero element $a \in L$ such that if $b \leq a$ for $b \in L$, then either $b = a$ or $b = 0$. An effect algebra L is said to be atomic if for any $a \in L \setminus \{0\}$ there exists an atom b of L such that $b \leq a$.

Let L be an effect algebra. Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in L. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \tag{2.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in effect algebras we conclude that (2.1) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$

if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n}, \tag{2.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n). \tag{2.3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in L is $\bigoplus$-*orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a $\bigoplus$-*sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n. \tag{2.4}$$

It is clear that two elements a and b of L are orthogonal, i.e. $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of L is $\bigoplus$-*orthogonal* iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$ is $\bigoplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\bigoplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\bigoplus$-orthogonal system $G = \{a_i\}_{i \in I}$ of L has a $\bigoplus$-*sum* in L, written as $\bigoplus_{i \in I} a_i$, iff in L there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_F \bigoplus_{i \in F} a_i, \tag{2.5}$$

where F runs over all finite subsets in I. In this case, we also write $\bigoplus G := \bigoplus_{i \in I} a_i$.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\bigoplus$-orthogonal, then the $\bigoplus$-sums defined by (2.4) and (2.5) coincide.

If (EAiv) is changed to

(OA) if $a \oplus a$ is defined, then $a = 0$ (consistency), we say that L is an *orthoalgebra* [10].

An *orthomodular poset* (OMP in abbreviation) is an orthoalgebra L such that if, for $a, b \in L$ we have $a \perp b$, then $a \vee b \in L$. If it is the case, then $a \vee b = a \oplus b$. In particular, if $c \leq d$, $c, d \in L$, then $d \ominus c = d \wedge c^{\perp} = (d^{\perp} \vee c)^{\perp}$.

A so-called Wright triangle [10] is the most simple case of orthoalgebra which is not an OMP.

An OMP L which is a lattice is said to be an *orthomodular lattice* (OML in abbreviation). An OMP L is said to be a σ-OMP if, for any sequence $\{a_n\}$ of mutually orthogonal elements of L, $\bigvee_n a_n \in L$, in this case, $\bigvee_n a_n = \bigoplus_n a_n$.

Example 2.1. Let $L(H)$ be the system of all closed subspaces of a real or complex Hilbert space H. Then $L(H)$ is a complete OML. It is used as the most important example of quantum mechanics models.

Example 2.2. Let S be a real or complex inner product space and denote by $E(S)$ the set of all splitting subspaces of S, i.e. of all subspaces $M \subseteq S$ such that $M + M^{\perp} = S$, where $M^{\perp} := \{x \in S : (x, y) = 0 \text{ for any } y \in M\}$. Then $E(S)$ is an OMP which is a σ-OMP iff S is complete [8], Thm 4.1.6.

Example 2.3. Let $\mathcal{A}$ be a von Neumann algebra of operators acting on a complex Hilbert space H. The system $P(\mathcal{A})$ of all orthogonal projections from $\mathcal{A}$ is a complete OML called the projection lattice of $\mathcal{A}$.

Example 2.4. The set $\mathcal{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is an effect algebra (which is not an orthoalgebra); a partial ordering $\leq$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = A \oplus B$ iff $(Ax, x) + (Bx, x) = (Cx, x)$, $x \in H$.

Example 2.5. Let the closed interval $[0, 1]$ be ordered by the natural ordering. Let g be any continuous, increasing mapping from $[0, 1]$ onto $[0, 1]$ such that $g(0) = 0$ and $g(1) = 1$ (called a generator). Define a partial binary operation $\oplus_g$ via

$$a \oplus_g b := g^{-1}(g(a) - g(b))$$

supposing $g(a) + g(b) \leq 1$. Then L with $0, 1$, and $\oplus_g$ is an effect algebra which is a distributive lattice and no orthoalgebra. In particular, if $g = id_{[0,1]}$, then $a \oplus_{id} b = a + b$.

A real valued mapping m on an orthoalgebra L is said to be a (finitely additive) *measure* if

$$m(a \oplus b) = m(a) + m(b), \; a, b \in L.$$

It is clear that $m(0) = 0$.

If for a mapping $m: \; L \to \mathbf{R}$ we have

$$m(\bigoplus_{i\in I} a_i) = \sum_{i\in I} m(a_i), \tag{2.6}$$

whenever $\bigoplus_{i\in I} a_i$ exists in L, m is said to be a *σ-additive* or *completely additive measure* if (2.6) holds for any countable or any index set I, respectively. If a measure m is positive, then

$$m(a) \leq m(b) \text{ whenever } a \leq b.$$

A positive measure m with $m(1) = 1$ is said to be a *state*, the system of all states on L is denoted by $\Omega(L)$. A measure m is said to be (i) *Jordan* if there are two positive measures m_1 and m_2 on L such that $m = m_1 - m_2$; (ii) *bounded* if $\sup\{|m(a)| : \; a \in L\} < \infty$. We denote by $J(L)$ and $W(L)$ the sets of all Jordan measures of bounded measures on L, respectively. It is evident that $\Omega(L) \subset J(L) \subseteq W(L)$.

Let m be a bounded measure on L. We define real valued mappings $m^+, m^-, |m|$ on L as follows

$$m^+(a) := \sup_{b\leq a} m(b), \quad m^-(a) := -\inf_{b\leq a} m(b), \quad \text{and} \quad |m| := m^+ + m^-,$$

for any $a \in L$. Then $m^+, m^-, |m|$ are said to be a *positive, negative* and *total variation*, respectively, of m.

Notice that m^- is equal to $(-m)^+$. We have the following simple properties of variations:

(i) $m^+(a) \leq m^+(b)$ whenever $a \leq b$ and $a, b \in L$.

(ii) m^+ is superadditive, i.e., if $\bigoplus_i a_i$ is defined in L, then $m^+(\bigoplus_i a_i) \geq \sum_i m^+(a_i)$.

(iii) (i) and (ii) holds also for m^- and $|m|$.

(iv) $|m(a)| \leq |m|(a)$ for any $a \in L$.

(v) If $a \in L$ is an atom, then
$m^+(a) = \max\{0, m(a)\}, m^-(a) = -\inf\{0, m(a)\}$, and $|m|(a) = |m(a)|$.

(vi) $m(a) = m^+(a) - m^-(a),\ a \in L$.

For any $m \in W(L)$, we denote by

$$||m||_s := \sup\{|m(a)| : a \in L\} \quad \text{and} \quad ||m||_v := |m|(1)$$

the *variation norm* and the *sup-norm* of m, respectively. According to [19], we have, for any $a \in L$,

$$|m|(a) = \sup_{b \leq a}(m(b) - m(a \ominus b)) leqno(2.7)$$

$$= 2 \sup_{b \leq a} |m(b)| - |m(a)| = 2 \sup_{b \leq a} m(b) - m(a).$$

3. Measures with Gleason property

Let L be a σ-OML, i.e. L is a σ-OMP which is also a lattice. Then L is also a σ-lattice. An *observable* of L is a mapping $x : \mathcal{B}(\mathbf{R}) \to L$, where $\mathcal{B}(\mathbf{R})$ is the Borel σ-algebra of the real line $\mathbf{R}$, such that (i) $x(\mathbf{R}) = 1$; (ii) $x(\mathbf{R} \setminus E) = x(E)^\perp$ for any $E \in \mathcal{B}(\mathbf{R})$; (iii) $x(\bigcup_{i=1}^\infty E_i) = \bigvee_{i=1}^\infty x(E_i)$ for any sequence $\{E_i\}_{i=1}^\infty$ in $\mathcal{B}(\mathbf{R})$. An observable x is *bounded* if there exits a compact set C such that $x(C) = 1$. The *spectrum* of x is the set $\sigma(x) := \bigcap\{C : C$ is compact, $x(C) = 1\}$. The *norm* of a bounded observable x is the expression $||x|| := \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Denote by $O(L)$ the set of all bounded observables on L. In particular, the *question observable* associated with any element $a \in L$, i.e. an observable q_a such that $q_a(\{0\}) = a^\perp$, $q_a(\{1\}) = a$, is a bounded observable on L. Two observables x and y are *compatible*, and we write $x \leftrightarrow y$, if there is a Boolean subalgebra of L containing both ranges $R(x)$ and $R(y)$ of x and y, where $R(x) = \{x(E) : E \in \mathcal{B}(\mathbf{R})\}$. For observables and calculus of compatible observables see [20].

If x is an observable and m is a measure on L, then $m_x : \mathcal{B}(\mathbf{R}) \to \mathbf{R}$ is a usual signed measure on $\mathcal{B}(\mathbf{R})$. The expectation of x in a state m is the expression

$$m(x) := \int_{\sigma(x)} \lambda \, dm_x(\lambda) \tag{3.1}$$

if the integral on the right-hand side of (3.1) exists and is finite.

A non-void system $\mathcal{M}$ of states on L is said to be *quite full* if, for $a, b \in L$, the inclusion $\{m \in \mathcal{M} : m(a) = 1\} \subseteq \{m \in \mathcal{M} : m(b) = 1\}$ implies $a \le b$. A *sum logic* is a σ-OMP L with a quite full system $\mathcal{S}(L)$ of all σ-additive states on L such that, for any pair of bounded observables x and y on L there is unique bounded observable $z \in O(L)$ such that

$$m(z) = m(x) + m(y), \ m \in \mathcal{S}(L),$$

and we write $z = x + y$. Then $O(L)$ is a real normed space.

If $m \in \mathcal{S}(L)$, then the mapping $\phi_m : O(L) \to \mathbf{R}$ such that $\phi_m(x) = m(x)$, $x \in O(L)$, is a bounded linear functional on $O(L)$ such that $||\phi_m|| = 1$. A measure $m \in W(L)$ is said to have a *Gleason property* if the functional $\phi_m : q_a \mapsto m(a)$, $a \in L$, can be extended to a bounded linear functional on $O(L)$.

Now we formulate Nikodým's boundedness theorem for measures with the Gleason property.

Theorem 3.1. *Let $\{m_\gamma : \gamma \in \Gamma\}$ be a family of bounded finitely additive measures having the Gleason property on the sum logic L for which $O(L)$ is a real Banach space. Let, for any $a \in L$, there exists a positive constant $K(a)$ such that the inequality*

$$|m_\gamma(a)| \le K(a) \tag{3.2}$$

holds for any $\gamma \in \Gamma$. Then there is a positive constant K such that the inequality

$$|m_\gamma(a)| \le K \tag{3.3}$$

holds for any $\gamma \in \Gamma$ and any $a \in L$.

Proof. Let x be an arbitrary but fixed bounded observable on L. Then the range $R(x) = \{x(E) : \; E \in \mathcal{B}(\mathbf{R})\}$ is a Boolean σ-subalgebra of L. The restriction $m_\gamma|R(x)$ of m_γ to $R(x)$ is a bounded finitely additive measure on $R(x)$ for any $\gamma \in \Gamma$. By [1], Thm 8, there exists a constant $K_x > 0$ such that

$$\sup\{|m_\gamma(a)| : a \in R(x),\ \gamma \in \Gamma\} \leq K_x. \tag{3.4}$$

Take a bounded function f_γ on $O(L)$ defined via $f_\gamma(q_a) = m_\gamma(a)$, $q_a \in O(L)$, and let

$$y = \sum_{i=1}^{n} \lambda_i q_{a_i} \tag{3.5}$$

be a bounded observable on L such that $\lambda_i \geq 0$ for $i = 1, \dots, n$ and $a_1, \dots a_n$ are mutually orthogonal elements from $R(x)$. Assume that $m_\gamma(a_1), \dots, m_\gamma(a_k)$ ≥ 0, and $m_\gamma(a_{k+1}), \dots, m(a_n) < 0$, and define $a = \bigvee_{i=1}^{k} a_i$ and $b = \bigvee_{i=k+1}^{n}$. An easy calculation gives us

$$\begin{aligned} |f_\gamma(y)| &= \sum_{i=1}^{n} \lambda_i m_\gamma(a_i) \leq \sum_{i=1}^{k} \lambda_i m_\gamma(a_i) - \sum_{i=k+1}^{n} \lambda_i m_\gamma(a_i) \\ &\leq \max_{1\leq i\leq n} \lambda_i(|m_\gamma(a)| + |m_\gamma(b)|) \leq 2\,K_x\,||y||. \end{aligned}$$

Any bounded observable y with $\sigma(y) \subseteq [0, \infty)$ and with the range in $R(x)$ is a uniform limit of a sequence $\{y_i\}$ of mutually compatible observables with ranges in $R(x)$ of the form (3.5). Therefore,

$$|f_\gamma(y)| \leq 2\,K_x\,||y||.$$

Since $x = x^+ - x^-$, where $\sigma(x^+), \sigma(x^-) \subseteq [0, \infty)$, $x^+ \leftrightarrow x^-$, and $R(x^+)$, $R(x^-) \subseteq R(x)$, we have $|f_\gamma(x^+)| \leq 2\,K_x\,||x^+|| \leq 2\,K_x\,||x||$, and $|f_\gamma(x^-)| \leq 2\,K_x\,||x^-|| \leq 2\,K_x\,||x||$, consequently,

$$|f_\gamma(x)| \leq 4\,K_x\,||x||. \tag{3.6}$$

Applying the Banach-Steinhaus principle of uniform boundedness to the Banach space $O(L)$, ([7] or [1] for an "elementary" proof), there exists a constant $K > 0$ such that

$$||f_\gamma|| \leq K \quad \text{for any} \quad \gamma \in \Gamma.$$

Finally,

$$|m_\gamma(a)| = |f_\gamma(q_a)| \leq ||f_\gamma|| \, ||q_a|| \leq K \cdot 1 = K,$$

which conclude the proof. □

Remark 3.1. (i) If $L = P(\mathcal{A})$, where $\mathcal{A}$ is a von Neumann algebra without I_2 as direct summand, then any bounded finitely additive measure has the Gleason property, [4], and, in addition, $O(P(\mathcal{A}))$ is a real Banach space, because any bounded observable on $P(\mathcal{A})$ uniquely corresponds to a Hermitian operator A from $\mathcal{A}$ and vice versa. Hence, Nikodým's boundedness theorem holds for any system of bounded finitely additive measures on $P(\mathcal{A})$. In particular, if $\mathcal{A} = B(H), \dim H \geq 3$, then $L(\mathcal{A})$ is isomorphic with $L(H)$, and Nikodým's boundedness theorem holds for bounded finitely additive measures on $L(H)$.

(ii) Any completely additive measure on $P(\mathcal{A})$ of a von Neumann algebra $\mathcal{A}$ without $I_n, n \geq 2$ as direct summand is by Dorofeev [5] bounded. Hence by [4], it has the Gleason property. Consequently, the result of Hamhalter [13] follows from Theorem 3.1.

(iii) If $L = L(H)$, where $\dim H = \infty$, then any completely additive measure m on $L(H)$ is bounded [6], [8], Thm 3.2.20. Then for m there is a unique Hermitian trace operator T on H such that the Gleason formula

$$m(M) = \mathrm{tr}(TP_M), \;\; M \in L(H) \tag{3.7}$$

holds, where P_M denotes the orthogonal projection from H onto M. Consequently, any completely additive measure m has the Gleason property, and the result from [9] follows also from Theorem 3.1

(iv) If L is a σ-algebra $\mathcal{S}$ of subsets of a set $\Omega \neq \emptyset$, then any bounded measure on $\mathcal{S}$ has the Gleason property [7], Thm 2.18-2.22, and Theorem 3.1 holds also for this case.

(v) Nikodým's boundedness theorem is valid also for a system of bounded finitely additive measures on the set of all effects, $\mathcal{E}(H)$, of a complex Hilbert space H. This follows from a simple fact that $L(H)$ can be naturally embedded into $\mathcal{E}(H)$ and applying the part (i).

We now generalize Theorem 3.1 for vector-valued measures.

Let X be a normed space and L an orthoalgebra. The mapping $\nu : L \to X$ is said to be a *finitely additive vector-valued measure* if

$$\nu(a \oplus b) = \nu(a) + \nu(b), \ a, b \in L.$$

ν is bounded if $\sup\{||\nu(a)|| : a \in L\} < \infty$.

Theorem 3.2. *Let $\{\nu_\gamma : \ \gamma \in \Gamma\}$ be a system of bounded finitely additive vector-valued measures with values in a Banach space X, defined on a sum logic L such that $O(L)$ is a real Banach space. Let any bounded measure on L has the Gleason property. Let, for any $a \in L$, there is a positive constant $K(a)$ such that the inequality*

$$||\nu_\gamma(a)|| \leq K(a)$$

holds for any $\gamma \in \Gamma$. Then there is a positive constant K such that the inequality

$$||\nu_\gamma(a)|| \leq K$$

holds for any $\gamma \in \Gamma$ and any $a \in L$.

Proof. Let ϕ be a linear continuous functional on X. Then, for any $\gamma \in \Gamma$, the function $m_\gamma^\phi(a) := \phi(\nu_\gamma(a))$, $a \in L$, is a complex-valued finitely additive measure on L. Separating the real and imaginary parts of $m_\gamma^\phi(a)$, we see that they are real-valued finitely additive measures on L having the Gleason property. Using Theorem 3.1, we can find a constant $K_\phi > 0$ such $|\phi(\nu_\gamma(a))| \leq K_\phi$ for any $\gamma \in \Gamma$ and any $a \in L$. Therefore, the set $\{\nu_\gamma(a) : \ a \in L, \gamma \in \Gamma\}$ is bounded in the weak topology. So this set is bounded in the norm topology of X, which concludes the proof. □

4. Regular measures

Let $\mathcal{K}$ be a non-empty set of an effect algebra L. A real-valued mapping m of L is said to be $\mathcal{K}$-*regular* if given $a \in L$ and given $\epsilon > 0$ there exists an element $b \in \mathcal{K}$ with $b \leq a$ such that

$$|m(b) - m(a)| < \epsilon.$$

For example, if $\mathcal{K} = L$, any $m \in \mathbf{R}^L$ is $\mathcal{K}$-regular.

It is evident that if m is a $\mathcal{K}$-regular mapping, then (i) $0 \in \mathcal{K}$; (ii) if a is an atom in L and $m(a) \neq 0$ and $m(0) = 0$, then $a \in \mathcal{K}$. We say that $\mathcal{K}$ is (i) *directed upwards* if, for any $b_1, b_2 \in \mathcal{K}$, there is $b \in \mathcal{K}$ such that $b_1, b_2 \leq b$; (ii) *densely upwards directed* in L if, for any $b_1, b_2 \in \mathcal{K}$ and any $a \in L$ with $b_1, b_2 \leq a$, there exists an element $b \in \mathcal{K}$ with $b \leq a$ such that $b_1, b_2 \leq b$; (iii) $\bigoplus$-*dense* in L if, for any $a \in L$, there exists a $\bigoplus$-orthogonal system $\{b_i\}_i$ in $\mathcal{K}$ such that $a = \bigoplus_i b_i$.

For example, if $\mathcal{K} = P(S)$ is the system of all finite-dimensional subspaces of an inner product space S, then $P(S)$ is a densely upwards directed system dense in $E(S)$.

Proposition 4.1. *Let m be a bounded $\mathcal{K}$-regular measure on L. Then m^+ and m^- are $\mathcal{K}$-regular mappings. If, in addition, $\mathcal{K}$ is densely directed upwards in L, then $|m|$ is $\mathcal{K}$-regular.*

Proof. From the definition of m^+ we have that given $\epsilon > 0$ and $a \in L$ we can find $b_1 \in L$ with $b_1 \leq a$ such that $m^+(a) < m(b_1) + \epsilon/2$. The $\mathcal{K}$-regularity of m entails the existence of $b \in \mathcal{K}$ with $b \leq b_1$ such that $|m(b_1) - m(b)| < \epsilon/2$. Hence

$$m^+(a) < m(b_1) + \epsilon/2 < m(b) + \epsilon \leq m^+(b) + \epsilon.$$

Since $m^- = (-m)^+$, m^- is $\mathcal{K}$-regular, too.

To prove the $\mathcal{K}$-regularity of $|m|$, we use the $\mathcal{K}$-regularity of both m^+ and m^- which has been just established. Given $a \in L$ and $\epsilon > 0$ there exist $b_1, b_2 \in \mathcal{K}$ with $b_1, b_2 \leq a$ and $m^+(a) < m^+(b_1) + \epsilon/2, m^-(a) < m^-(b_2) + \epsilon/2$. For b_1, b_2 there is an element $b \in \mathcal{K}$ with $b \leq a$ such that $b_1, b_2 \leq b$. Then

$$\begin{aligned} |m|(a) &= m^+(a) + m^-(a) < m^+(b_1) + m^-(b_2) + \epsilon \\ &\leq m^+(b) + m^-(b) + \epsilon = |m|(b) + \epsilon. \square \end{aligned}$$

Proposition 4.2. *Let $R_{\mathcal{K}}(L)$ be the set of all bounded $\mathcal{K}$-regular measures on an effect algebra L. Then* (i) *the null measure belongs to $R_{\mathcal{K}}(L)$;* (ii) *if $m \in R_{\mathcal{K}}(L)$ and $\alpha \in \mathbf{R}$, then $\alpha m \in R_{\mathcal{K}}(L)$;* (iii) *if $\mathcal{K}$ is densely upwards directed in L, then $m_1, m_2 \in R_{\mathcal{K}}(L)$ imply $m_1 + m_2 \in R_{\mathcal{K}}(L)$.*

Proof. (i) and (ii) are evident. To prove (iii), we have that given $a \in L$ and given $\epsilon > 0$ there exist two elements $b_1, b_2 \in \mathcal{K}$ with $b_1, b_2 \leq a$ such that

$|m_i|(a) < |m_i|(b_i) + \epsilon/2$ for $i = 1, 2$. For any $b \in \mathcal{K}$ with $b_1, b_2 \le b \le a$, we have

$$|m_1|(a) + |m_2|(a) < |m_1|(b_1) + |m_2|(b_2) + \epsilon \le |m_1|(b) + |m_2|(b) + \epsilon.$$

Hence

$$\begin{aligned} |m_1(a \ominus b) + m_2(a \ominus b)| &\le |m_1|(a \ominus b) + |m_2|(a \ominus b) \le |m_1|(a) - |m_1|(b) \\ &+ |m_2|(a) - |m_2|(b) < \epsilon, \end{aligned}$$

when we have used the superadditivity of total variations. □

Remark 4.1. If $\mathcal{K}$ is $\bigoplus$-dense and m is a completely additive measure, then m is $\mathcal{K}$-regular. The converse is not true, in general. For example, if S is an inner product space, $\mathcal{K} := P(S)$ is the system of all finite-dimensional subspaces of S, then $P(S)$ is $\bigoplus$-dense in $E(S)$ (see Example 2.2). By [8], Thm 4.3.4, for any Jordan $P(S)$-regular measure on $E(S)$, there exists a unique Hermitian trace operator T on the completion $\overline{S}$ of S such that

$$m(M) = \mathrm{tr}(TP_{\overline{M}}), \quad M \in E(S), \tag{4.1}$$

where $P_{\overline{M}}$ denotes the orthogonal projection from $\overline{S}$ onto the completion $\overline{M}$ of M. Conversely, the right-hand side of (4.1) determines a Jordan $P(S)$-regular measure on $E(S)$. But if S is incomplete, then $E(S)$ has no non-zero Jordan completely additive measure [8], Thm 4.2.3.

Denote by $\mathcal{F}(L)$ the system of all $a \in L$ such that there is a finite $\bigoplus$-orthogonal system F of atoms in L with $a = \bigoplus\{b \in F\}$; if $F = \emptyset$, we define $\bigoplus \emptyset := 0$. The elements of $\mathcal{F}(L)$ are said to be *finite*.

For example, in the Wright triangle $\mathcal{F}(L) = L$.

Let $a \in L \setminus \{0\}$, then $L_a = \{b \in L : b \le a\}$ can be organized in a natural way into an effect algebra with the greatest element a.

Lemma 4.1. *Let m be a $\mathcal{K}$-regular bounded measure on an effect algebra L. Then, for any $a \in L$, there exists an element $b \in \mathcal{K}$ with $a \le b$ such that*

$$|m(b)| > \frac{1}{4}|m|(a). \tag{4.2}$$

Proof. By (2.7) we have

$$|m|(a) = 2\sup_{b\le a}|m(b)| - |m(a)| = 2\sup_{b\le a} m(b) - m(a).$$

The $\mathcal{K}$-regularity of m implies

$$|m|(a) = 2\sup_{b\le a, b\in\mathcal{K}} m(b) - m(a) = 2\sup_{b\le a, b\in\mathcal{K}} |m(b)| - |m(a)|.$$

If $|m|(a) = 0$, the statement (4.2) is evident. If $|m|(a) > 0$, put $\epsilon = |m|(a)/2$. Then there exists an element $b \in \mathcal{K}$ with $b \le a$ such that

$$|m|(a) < 2|m(b)| + \epsilon - |m(a)| \le 2|m(b)| + |m|(a)/2$$

which gives (4.2). □

We now present some partial results concerning the Nikodým boundedness theorem for regular measures on effect algebras. First of all we formulate simple assertions:

Lemma 4.2. *Let $\sum_{n\in\mathbf{N}}|x_n| < \infty$, where $\{x_n\}_{n\in\mathbf{N}}$ is a sequence of real numbers. Then, there is a subset N^0 of* $\mathbf{N}$ *such that*

$$\left|\sum_{n\in N^0} x_n\right| \ge \frac{1}{2}\sum_{n\in\mathbf{N}}|x_n|. \tag{4.3}$$

In addition, there is a finite subset N' of $\mathbf{N}$ *such that*

$$\left|\sum_{n\in N'} x_n\right| \ge \frac{1}{4}\sum_{n\in\mathbf{N}}|x_n|. \tag{4.4}$$

Proof. Denote by $N^+ = \{n \in \mathbf{N}: \ x_n \ge 0\}$ and $N^- = \{n \in \mathbf{N}: \ x_n < 0\}$, and put $S^+ = |\sum_{n\in N^+} x_n| < \infty$, $S^- = |\sum_{n\in N^-} x_n| < \infty$. Then either $S^+ \ge S^-$ or $S^+ < S^-$. In the first case we define $N^0 := N^+$ and in the second one $N^0 := N^-$. Then

$$\left|\sum_{n\in N^0} x_n\right| = \max\{S^+, S^-\} \ge \frac{1}{2}(S^+ + S^-) = \frac{1}{2}\sum_{n\in\mathbf{N}}|x_n|.$$

In addition, it is evident that there exists a finite subset N' of N^0 such that

$$\left|\sum_{n\in N'} x_n\right| \ge \frac{1}{2}\left|\sum_{n\in N^0} x_n\right|.$$

Consequently, (4.4) holds. □

Proposition 4.3. *Let $\{m_\gamma : \gamma \in \Gamma\}$ be a family of bounded finitely additive measures on an effect algebra L. Let, for any $a \in L$, there exists a positive constant $K(a)$ such that (3.2) holds for $\gamma \in \Gamma$. Let $\mathcal{K}$ be a non-void system of L and let $\{b_j\}_{j\in J}$ be at most a countable $\bigoplus$-orthogonal system of elements from $\mathcal{K}$ such that, for any $I \subseteq J$, $\bigoplus_{j\in I} b_j$ exists in L. Then there exists a positive constant $K_{\{b_j\}}$ such*

$$\sup_{\gamma\in\Gamma} \sum_j |m_\gamma(b_j)| \le K_{\{b_j\}}.$$

Proof. If J for $\{b_j\}_{j\in J}$ is finite, the statement is evident. Suppose thus $J = \{1, 2, \ldots\}$. Define a measurable space $(J, 2^J)$ and measures μ_γ on 2^J via

$$\mu_\gamma(B) := m_\gamma\Big(\bigoplus_{j\in B} b_j\Big), \quad B \in 2^I.$$

Any μ_γ is a bounded finitely additive measure on 2^J. Using [1], Thm 8, there is a positive constant $K_{\{b_j\}}$ such that the inequality

$$|\mu_\gamma(B)| \le K_{\{b_j\}}/2$$

holds for any $\gamma \in \Gamma$ and any $B \in 2^J$. For any disjoint sets $B_1, \ldots B_n \in 2^J$, by Lemma 4.5, for any $\gamma \in \Gamma$ there is a finite subset $S_\gamma \subseteq \{1, \ldots, n\}$ such

$$K_{\{b_j\}}/2 \ge \Big|\sum_{j\in S_\gamma} \mu_\gamma(B_j)\Big| \ge \frac{1}{2}\sum_{j=1}^{n} |\mu_\gamma(B_j)|$$

so that $\sum_{j=1}^{n} |\mu_\gamma(B_j)| \le K_{\{b_j\}}$. Consequently, $\sum_{j=1}^{\infty} |\mu_\gamma(B_j)| \le K_{\{b_j\}}$ holds for any sequence of mutually disjoint subsets of J. In particular,

$$\sum_{j=1}^{\infty} |m_\gamma(b_j)| = \sum_{j=1}^{\infty} |\mu_\gamma(\{j\})| \le K_{\{b_j\}}.$$

□

Proposition 4.4. *Let $\{m_\gamma : \gamma \in \Gamma\}$ be a family of $\mathcal{K}$-regular, bounded, finitely additive measures on an effect algebra L. Let, for any $a \in L$, there exists a positive constant $K(a)$ such that (3.2) holds for $\gamma \in \Gamma$. Let any sequence of $\bigoplus$-orthogonal elements from $\mathcal{K}$ has the sum in L. Let $\{a_j\}_{j\in J}$*

be at most a countable $\bigoplus$-orthogonal system of elements from L. Then there exists a positive constant $K_{\{a_j\}}$ such

$$\sup_{\gamma\in\Gamma} |m_\gamma(a_j)| \le K_{\{a_j\}}.$$

Proof. If the sequence $\{a_j\}$ is finite, the statement of the Proposition trivially holds. Let now $\{a_j\}$ be an infinite sequence and suppose that the statement of the Proposition is false. Then for $K = 4$ there exist an integer k_1 and a measure m_{γ_1} such that

$$\sum_{j=1}^{k_1} |m_{\gamma_1}(a_j)| > 4.$$

Then

$$\sup_{\gamma\in\Gamma} \sum_{j=k_1+1}^{\infty} |m_\gamma(a_j)| = \infty$$

in view of

$$\sup_{\gamma\in\Gamma} \sum_{j=1}^{k_1} |m_\gamma(a_j)| \le k_1 \max\{K(a_1),\dots,K(a_{k_1})\} < \infty.$$

Continuing by induction, we find an increasing sequence of integers $0 =: k_0 < k_1 < k_2 < \cdots$ and a sequence of measures $\{m_{\gamma_i}\}$ such that, for any i,

$$\sum_{j=k_{i-1}+1}^{k_i} |m_{\gamma_i}(a_j)| > 4\,i.$$

Put $J_i := \{k_{i-1}+1,\dots,k_i\}$. For any integer i and any $j \in J_i$, there exists, by Lemma 4.1, an element $b_j \in \mathcal{K}$ with $b_j \le a_j$ such that

$$|m_{\gamma_i}(b_j)| \ge \frac{1}{4}|m_{\gamma_i}(a_j)|. \tag{4.5}$$

It is simple to show by induction that $\{b_j\}$ is $\bigoplus$-orthogonal. By Proposition 4.3, $\sup_{\gamma\in\Gamma}\sum_{j=1}^{\infty} |m_\gamma(b_j)| < \infty$ which is a contradiction to (4.5), and the Proposition is proved. □

Proposition 4.5. *Let $\{m_\gamma : \gamma \in \Gamma\}$ be a family of $\mathcal{K}$-regular, bounded, finitely additive measures on an effect algebra L. Let, for any $a \in L$, there exists a positive constant $K(a)$ such that (3.2) holds for $\gamma \in \Gamma$. Let, for any $b \in \mathcal{K}$, $\sup_{\gamma\in\Gamma} |m_\gamma|(b) < \infty$, and let any sequence of $\bigoplus$-orthogonal elements from $\mathcal{K}$ has the sum in L. Let $\{a_j\}$ be at most a countable $\bigoplus$-orthogonal system from $\mathcal{K}$. Then there exists a positive constant $K_{\{a_j\}}$ such*

$$\sup_{\gamma\in\Gamma} |m_\gamma|(a_j) \le K_{\{a_j\}}.$$

Proof. When we change a_j and b_j from the proof of Proposition 4.4 to b_j and $b'_j \in \mathcal{K}$, and use the similar ideas, we obtain the assertion in question. □

Remark 4.2 (i) If, in addition, any $|m_\gamma|$ is additive in Proposition 4.5, the Nikodým boundedness theorem is true.

(ii) If L is a σ-algebra, then the total variation of a bounded measure is additive.

Example 4.1 Let $L = [0, 1]$ and $\oplus$ be the usual addition in $[0, 1]$. Let $\mathbf{Q}_1$ be the set of all rational numbers in $[0, 1]$. Then:

(i) If m is a measure on $[0, 1]$, then $m(q) = q\,m(1)$ for any $q \in \mathbf{Q}_1$, and $m(qt) = q\,m(t)$ for any $q \in \mathbf{Q}_1$ and any $t \in [0, 1]$ with $qt \in [0, 1]$.

(ii) Any Jordan measure m is either positive or negative, and it has the form

$$m(t) = t\,m(1), \quad t \in [0, 1], \tag{4.6}$$

consequently, it is completely additive.

(iii) The following assertions are equivalent:

(a) m is $\mathbf{Q}_1$-regular.

(b) m is σ-additive.

(c) m is completely additive.

(d) m is a Jordan measure.

(e) m is continuous in $[0, 1]$.

(f) m is bounded.

(g) m is of the form (4.6).

(iv) Let ψ be a discontinuous additive functional on $\mathbf{R}$, [12], then $m(t) := \psi(t),\ t \in [0,\,1]$, is an unbounded measure on $[0,\,1]$.

(v) Let m be an X-valued measure, where X is a Banach space. Then the following assertions are equivalent: (va) m is bounded; (vb) m is continuous; (vc) m has the form (4.6).

Proof. We prove only some of above statements. If m is positive, the formula (4.6) follows easily from the monotonicity of m.

(a) $\Rightarrow$ (g): The $\mathbf{Q}_1$-regularity of m implies that given $t \in (0,\,1)$ there is an increasing sequence of natural numbers, $\{q_n\}$, with $q_n \leq t$ such that $m(t) = \lim_n m(q_n) = \lim_n q_n m(1) = t_0 m(t)$, where $t_0 = \lim_n q_n$. Then $m(t)$ and $m(1)$ have the same sign, so that m is either positive or negative.

(f) $\Rightarrow$ (e): Let m be a bounded measure on $[0,\,1]$. Take a finite-dimensional Hilbert space H_n, $\dim H_n \geq 2$, and let $\mathcal{S}(H_n)$ be a unite sphere in H_n, that is, $\mathcal{S}(H_n) = \{x \in H_n :\ ||x|| = 1\}$. Fix a unit vector $e \in H_n$. Then the mapping $x \mapsto |(e,x)|^2$, $x \in \mathcal{S}(H_n)$, maps $\mathcal{S}(H_n)$ onto the interval $[0,\,1]$. Indeed, if e_1 is a unit vector in H_n which is orthogonal to e, then for unit vectors $x_\phi := \cos\phi\, e + \sin\phi\, e_1$, $\phi \in [0,\,\pi/2]$, we have $|(e,x_\phi)|^2 = \cos^2\phi$.

Define the mapping $f : \mathcal{S}(H_n) \to \mathbf{R}$ via $f(x) := m(|(e,x)|^2)$, $x \in \mathcal{S}(H_n)$. Then f is a frame function in H_n, i.e., if $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are two orthonormal bases in H_n, then $\sum_{i=1}^n f(x_i) = \sum_{i=1}^n f(y_i)$. The boundedness of m entails the boundedness of f. Using the Gleason theorem for finite-dimensional Hilbert spaces, [11], [8], Thm 3.2.15, f is continuous.

If now $t_n \to t$ in $[0,\,1]$, then there exist ϕ_n, ϕ in $[0,\,\pi/2]$ such that $\phi_n \to \phi$, and $t_n = |(e,x_{\phi_n})|^2 \to t = |(e,x_\phi)|^2$, so that $m(t_n) \to m(t)$ because $x_{\phi_n} \to x_\phi$. Therefore, m is continuous.

(v) Let m be a bounded vector-valued measure. Take an arbitrary continuous linear functional ψ on X. Then $\psi \circ m$ is a bounded complex-valued measure on $[0,\,1]$. Assuming separately the real and imaginary part of $\psi\circ m$, we see that $\psi \circ m$ is continuous. Hence by (4.6), $(\psi \circ m)(t) = t(\psi \circ m)(1) = \psi(tm(t))$ for any $t \in [0,\,1]$. Because all bounded functionals from the dual X^* of X separate the points of X, m has the form (4.6).

All other implications are simple. □

It is worth to notice that the total variation of any bounded measure on [0, 1] is additive. In addition, the Nikodým boundedness theorem (as well as the Nikodým convergence theorem) holds for any system of bounded measures on [0, 1], and, in addition, all conditions in (i) of Remark 4.2 and Proposition 4.5 are satisfied.

5. Regular measures on inner product spaces

In the present section we deal with measures on the set $E(S)$ of all splitting subspaces of a real or complex inner product space S (see Example 2.2). Before that we introduce some useful notions according to [19].

Let $m = m_1 - m_2$ be a Jordan measure on an effect algebra L, where m_1 and m_2 are positive measures on L; we say that m_1 and m_2 have (i) the *Hahn property* for m if there is an element $b \in L$ such that $m_1(b^\perp) = 0 = m_2(b)$; (ii) the *approximative Hahn property* for m if given $\epsilon > 0$, there exists an element $b \in L$ with $m_1(b^\perp), m_2(b) < \epsilon$; (iii) the *uniform approximative Hahn property* for m in L if given $a \in L$ and given $\epsilon > 0$ there is an element $b \in L$ with $b \le a$ such that $m_1(a \ominus b), m_2(b) < \epsilon$.

For example, for any bounded measure m on $L = [0, 1]$, there exists two positive measures m_1, m_2 with $m = m_1 - m_2$ which have the uniform approximative Hahn property for m on L (see Example 4.1).

Let Δ be a non-void convex subset of $\Omega(L)$, i.e., if $m_1, m_2 \in \Delta$, then $\lambda m_1 + (1-\lambda) m_2 \in \Delta$ for any $\lambda \in [0, 1]$. Let $J(\Delta)$ be the set of all Jordan measures on L generated by Δ, i.e., of all $m \in J(L)$ such that $m = s\,m_1 - t\,m_2$, where $s, t \in \mathbf{R}_+$, $m_1, m_2 \in \Delta$. For any $m \in J(\Delta)$, we define the *base norm* on $J(\Delta)$, $||\cdot||_\Delta$, via

$$(5.1) \qquad ||m||_\Delta = \inf\{s + t : \ m = s\,m_1 - t\,m_2, \ s, t \in \mathbf{R}_+, \ m_1, m_2 \in \Delta\}.$$

Then

$$||m||_s \le ||m||_v \le ||m||_\Delta.$$

Rüttimann [19], Thm 4.3 proved that if $m \in J(\Delta)$ has a Jordan decomposition $m = m_1 - m_2$, where m_1, m_2 are positive measures from $J(\Delta)$ such that m_1 and m_2 have the approximative Hahn property for m, then

$$(5.2) \qquad ||m||_\Delta = ||m||_v.$$

Proposition 5.1. *Let m_1 and m_2 be two positive measures and m be a bounded measure with $m = m_1 - m_2$ on an effect algebra L. The following statements are equivalent:*

(i) *m_1 and m_2 have the approximative Hahn property for m.*

(ii) $|m|(1) = m_1(1) + m_2(1)$.

(iii) $m^+(1) = m_1(1),\ m^-(1) = m_2(1)$.

Moreover, m_1 and m_2 have the uniform approximative Hahn property for m on L iff $|m|(a) = m_1(a) + m_2(a),\ a \in L \iff m^+(a) = m_1(a), m^-(a) = m_2(a),\ a \in L$.

Proof. It is clear that $m^+(a) \le m_1(a), m^-(a) \le m_2(a)$ for any $a \in L$, so that $|m|(1) \le m_1(1) + m_2(1)$. If m_1 and m_2 have the approximative Hahn property for m, given $\epsilon > 0$, there exists an element $b \in L$ with $m_1(b^\perp), m_2(b) < \epsilon/2$. Then

$$\begin{aligned}&m_1(b^\perp) + m_2(b) < \epsilon,\\ &m_1(1) - m_1(b) + m_2(b) < \epsilon,\\ &m^+(1) \le m_1(1) < m(b) + \epsilon \le m^+(1) + \epsilon,\end{aligned}$$

so that $m^+(1) = m_1(1)$. Since $m^-(1) = (-m)^+(1),\ m^-(1) = m_2(1)$, and $|m|(1) = m_1(1) + m_2(1)$.

Let now (ii) hold. From (2.7) we have that given $\epsilon > 0$ there exists an element $b \in L$ such that $|m|(1) < 2\,m(b) - m(1) + 2\,\epsilon$. Then $m_1(1) + m_2(1) < 2\,m_1(b) - 2\,m_2(b) - m_1(1) + m_2(1) < 2\,\epsilon$. So that, $m_1(b^\perp) + m_2(b) < \epsilon$, which means that $m_1(b^\perp), m_2(b) < \epsilon$.

Similarly, we prove the second part of the Proposition.□

Theorem 5.1. *Any bounded $P(S)$-regular measure m on $E(S)$,* $\dim S \ge 3$, *is a Jordan one with $m = m_1 - m_2$ such that m_1 and m_2 are $P(S)$-regular positive measures on $E(S)$, and they have the approximative Hahn property for m.*

Proof. Due to [8], Ex 4.4.5.24, any bounded $P(S)$-regular measure m on $E(S)$ is a Jordan one, and by [8], Thm 4.2.3, any Jordan $P(S)$-regular measure on $E(S)$ is in a one-to-one correspondence with the set of all Hermitian trace operators T on the completion $\overline{S}$ of S via (4.1). Denote by $m_1(M) := \mathrm{tr}(T^+ P_{\overline{M}})$ and $m_2(M) := \mathrm{tr}(T^- P_{\overline{M}})$, $M \in E(S)$, where T^+ and T^- are the positive and negative part of T, $T = T^+ - T^-$.

Since both T^+ and T^- are Hermitian trace operators on $\overline{S}$, then

$$T^+ = \sum_i \lambda_i(\cdot, x_i)\, x_i, \qquad T^- = \sum_j \mu_j(\cdot, y_j)\, y_j,$$

where $\lambda_i > 0, \mu_j > 0, \sum_i \lambda_i < \infty, \sum_j \mu_j < \infty$ and $\{x_i\} \cup \{y_j\}$ is an orthonormal system in $\overline{S}$.

Given $\epsilon > 0$ there is an integer n such that $\sum_{i>n} \lambda_i < \epsilon$, and without loss of generality we can assume that $n > \epsilon$. Define

$$K_1 := \begin{cases} 1 & \text{if } ||T|| \le 1, \\ 1/||T|| & \text{if } ||T|| > 1, \end{cases} \qquad K_2 := \begin{cases} 1 & \text{if } \sum_j \mu_j \le 1, \\ 1/\sum_j \mu_j & \text{if } \sum_j \mu_j > 1. \end{cases}$$

According to the proof of Theorem 4.1.2 from [8], we can find an orthonormal system $\{h_1, \ldots, h_n\}$ in S such that $||h_k - x_k|| < \epsilon\, K_1 K_2/(4n)$ for $k = 1, \ldots, n$. Then for any $k = 1, \ldots, n$,

$$\begin{aligned} & |(T^+h_k, h_k) - \lambda_k| = |(T^+h_k, h_k) - (T^+x_k, x_k)| \le |(T^+(h_k - x_k), x_k)| \\ + \ & |(T^+h_k, h_k - x_k)| \le 2\,||T^+||\;||h_k - x_k|| \le 2\,||T||\;||h_k - x_k|| \\ < \ & \epsilon\, K_2/2n \le \epsilon\,/2n. \end{aligned}$$

Thus, for $N = \mathrm{sp}(h_1, \ldots, h_n) \in E(S)$, we have

$$\begin{aligned} m_1(N^\perp) &= \sum_i \lambda_i - m_1(N) = \sum_{i>n} \lambda_i + \sum_{i=1}^{n} (\lambda_i - m_1(\mathrm{sp}(h_i)) \\ &= \sum_{i>n} \lambda_i + \sum_{i=1}^{n} (\lambda_i - (T^+h_i, h_i)) < \epsilon/2 + n\,\epsilon/2n = \epsilon. \end{aligned}$$

On the other hand,

$$m_2(N) = \sum_{k=1}^{n} m_2(\mathrm{sp}(h_k)) = \sum_{k=1}^{n} \sum_j \mu_j\, |(y_j, h_k)|^2$$

$$\begin{aligned} &= \sum_{k=1}^{n}\sum_{j}\mu_j|(y_j,h_k-x_k)|^2 \leq \sum_{k=1}^{n}\sum_{j}\mu_j\,||y_j||^2\,||h_k-x_k||^2 \\ &\leq \sum_{k=1}^{n}\sum_{j}\mu_j\,\epsilon^2K_1^2K_2^2/16n^2 = \epsilon^2\sum_{j}\mu_jK_2^2/16n < \epsilon^2/16n < \epsilon.\square \end{aligned}$$

The following statement follows easily from (2.7) and Theorem 5.1.

Corollary 5.1. *Let T be an arbitrary Hermitian trace operator on $\overline{S}$ and let m_T be a $P(S)$-regular measure on $E(S)$ determined by (4.1). Then*

$$\begin{aligned} \max\{tr(T^+),\, tr(T^-)\} &= ||m_T||_s \leq ||m_T||_v = ||m_T||_{\Delta(S)} = tr(|T|) \\ &= 2\,||m_T||_s - |tr(T)| = 2\,tr(|T|) - tr(T). \end{aligned}$$

where $\Delta(S)$ is the set of all $P(S)$-regular states on $E(S)$.

If $S = H$ is a Hilbert space, then any $P(H)$-regular measure m on $E(H) = L(H)$, $\dim H \geq 3$, is completely additive and vice versa, and naturally, $m_1 := m_{T^+}$ and $m_2 := m_{T^-}$, which are defined by (4.1) for T^+ and T^-, have the approximative Hahn property as well as the Hahn property for m; the subspace M generated by the proper vectors of T^+ has he property $m_1(M^\perp) = 0 = m_2(M)$. If S is an incomplete inner product space, it can happen that m_1 and m_2 have no Hahn property as we shall see in the following example. We recall that by dimension of S we understand the orthogonal dimension of S, i.e., the cardinality of any maximal orthogonal system is S.

Example 5.1. Let S be an incomplete inner product space of countable dimension.[2] Since an inner product space is complete iff any maximal orthonormal system (MONS for short) in S is an orthonormal basis (ONB for short), there is a MONS $\{e_n\}_{n=1}^{\infty}$ in S which is not ONB, consequently, there exists a unit vector $e \in \overline{S}\setminus S$ which is orthogonal with any e_n. Define $m_1 = ||P_{\overline{M}}e||^2$, $m_2 = \sum_{n=1}^{\infty} n^{-2}||P_{\overline{M}}e_n||^2$, $M \in E(S)$. Then m_1 and m_2 are $P(S)$-regular measures on $E(S)$ which have no Hahn property for the P(S)-regular measure $m = m_1 = m_2$.

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[2]For example, any separable inner product space is of countable dimension.

Indeed, let $m_1(M^\perp) = 0 = m_2(M)$. Hence, any $e_n \perp M$, so that M is the null space while $e \in \overline{M}$ which is a contradiction.

The Hahn property gives the following completeness criterion.

Theorem 5.2. *An inner product space S is complete if and only if, for any $P(S)$-regular measure $m := m_T$ on $E(S)$, the $P(S)$-regular measures $m_1 := m_{T^+}$ and $m_2 := m_{T^-}$ defined by* (4.1) *have the Hahn property.*

Proof. If S is complete, the statement is evident.

For the converse statement, choose a non-zero vector $x \in \overline{S}$. We claim to show that $x \in S$. Obviously, there exists $z \in S$ such that $z \not\perp x$. Then, for

$$y = x - ||x||^2/(z,x)\, z,$$

we have $y \perp x$ and $0 \neq y \in \overline{S}$. According to [2] or [8], Thm 4.1.2, there exist two sequences of vectors in S, $\{x_i\}$ and $\{y_j\}$, such that $x_i \perp y_j$ for all i and all j, and $x_i \to x$, $y_j \to y$. Applying the Gram-Schmidt orthogonalization process to $\{x_i\}$ and $\{y_j\}$, we find two mutually orthogonal sequences $\{f_i\}$ and $\{g_j\}$.

Define two positive Hermitian trace operators, $T_1 = \sum_i i^{-2}(\cdot, f_i)\, f_i$ and $T_2 = \sum_j j^{-2}(\cdot, g_j)\, g_j$, and put $T = T_1 - T_2$. Then $T^+ = T_1$ and $T^- = T_2$. Let m, m_1 and m_2 be $P(S)$-regular measures on $E(S)$ determined by T, T_1 and T_2, respectively, via (4.1). By the assumptions of Theorem, m_1 and m_2 have the Hahn property for m, so that, there exists an element $M \in E(S)$ such that $m_1(M^\perp) = 0 = m_2(M)$. Therefore, $f_i \in M$ and $g_i \perp M$ for every i, so that $x_i \in M$ and $y_i \perp M$ for any i, and, consequently, $x \in \overline{M}$ and $y \in \overline{M^\perp}$.

Denote by $P_{\overline{M}}$ the orthogonal projection from $\overline{S}$ onto $\overline{M}$. Then $0 = P_{\overline{M}}y = P_{\overline{M}}x - P_{\overline{M}}z_0$, where $z_0 = ||x||^2/(z,x)\, z \in S$, so that $P_{\overline{M}}z = z_1$, when $z_0 = z_1 + z_2$, $z_1 \in M$, $z_2 \in M^\perp$, and this gives $x = z_1 \in S$. □

When we would like to prove the Nikodým boundedness theorem for a family of bounded $P(S)$-regular measures on $E(S)$ applying some of propositions from Section 4, we have the following troubles:

(i) If $\mathcal{K} := P(S)$, then a $\bigoplus$-orthogonal system from $\mathcal{K}$ has the sum in $E(S)$ iff S is complete [8], Thm 4.1.6.

(ii) We do not know whether $\sup_\gamma |m_\gamma|(M) < \infty$ for any $M \in P(S)$.

Anyway, we have a partial result concerning Nikodým's boundedness theorem for incomplete inner product spaces. To present it, we introduce the following notion.

We say that an inner product space S has a *K-property* [1], [15] iff from any sequence $\{x_n\}_n$ from S with $\|x_n\| \to 0$ when $n \to \infty$ we can choose a subsequence $\{x_{n_i}\}_i$ of $\{x_n\}_n$ such that $\sum_{i=1}^j x_{n_i}$ converges to some element x in S when $j \to \infty$. Any Hilbert space has the K-property, but not all inner product spaces have the K-property as we see in the following example.

Example 5.2. Let H be a separable inner product space with an ONB $\{e_n\}_{n=1}^\infty$. Put $f = \sum_{n=1}^\infty n^{-1} e_n$ and define $S := \mathrm{sp}(f, e_2, e_3, \ldots)$. Then S is an incomplete inner product space, and $1/n\, e_n \to 0$ but there is no subsequence of $\{1/n\, e_n\}$ whose series is convergent to some element in S.

Kliś [14] has presented an example of incomplete inner product space which has the K-property.

Proposition 5.2. *Let S be an inner product space with the K-property. Let $\{m_{T_\gamma} : \gamma \in \Gamma\}$ be a family of $P(S)$-regular measures on $E(S)$ determined by a family of Hermitian trace operators $\{T_\gamma : \gamma \in \Gamma\}$ on $\overline{S}$ via* (4.1). *Let for any unit vector $x \in S$ there is a constant $K_x > 0$ such that*

$$\sup_{\gamma\in\Gamma} |m_{T_\gamma}(sp(x))| \le K_x, \tag{5.3}$$

then, for any integer n, there is a constant $K_n > 0$ such that, for any $M \in E(\overline{S})$, $\dim M \le n$,

$$\sup_{\gamma\in\Gamma} m_{|T_\gamma|}(M) \le K_n.$$

In particular,

$$\sup_{\gamma\in\Gamma} |m_{T_\gamma}|(M) \le K_n$$

for any at most n-dimensional subspace M of S.

Proof. For any unit vector $x \in S$, (5.3) entails $|(T_\gamma x, x)| \le K_x$ for any $\gamma \in \Gamma$. Using the polarization form, for any $x, y \in S$, there is a positive constant

K_{xy} such that $|(T_\gamma x, y)| \leq K_{xy}$ for any $\gamma \in \Gamma$. Since S has the K-property, the uniform boundedness principle holds for $\{T_\gamma : \gamma \in \Gamma\}$ on S [1], Thm 4.2, [15], and therefore, there exists a positive constant K such that, for all $x, y \in S$, $||x||, ||y|| = 1$,

$$\sup_{\gamma \in \Gamma} |(T_\gamma x, x)| \leq K. \tag{5.4}$$

Then (5.4) holds for all unit vectors $x, y \in \overline{S}$, so that, $\sup_{\gamma\in\Gamma} ||T_\gamma|| \leq K$. Since $||T_\gamma^+||, ||T_\gamma^-|| \leq ||T||$, we have $\sup_{\gamma\in\Gamma} ||\,|T_\gamma|\,|| \leq 2\,K$.

If now $M \in E(\overline{S})$, $\dim M \leq n$, then for any ONB $\{e_i\}_{i=1}^{\dim M}$, we have

$$\sup_{\gamma\in\Gamma} \operatorname{tr}(|T_\gamma| P_{\overline{M}}) = \sup_{\gamma\in\Gamma} \sum_{i=1}^{\dim M} (|T_\gamma| e_i, e_i) \leq 2\,nK. \square$$

Finally, we present an open problem.

Problem 5.1 (i) Does the Nikodým boundedness theorem hold for a family of bounded $P(S)$-regular measures on $E(S)$?

(ii) Does the Nikodým convergence theorem hold for a family of bounded $P(S)$-regular measures on $E(S)$? (See also [8], Pro 4.3.15.)

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Received by the editors January 10, 1996.