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GENERALIZED MOMENT PROBLEM IN VECTOR LATTICES

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Abstract

We present a moment problem in the context of vector lattices

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1. Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k dg(t), \qquad k \in N.$$

are called the moments of g. A sequence of real numbers $(a_n, n \in N)$ is said to give a solution of the moment problem if there exists a function g of bounded variation on I such that

$$a_k = \int_0^1 t^k dg(t)$$

for $k \in N$.

For every sequence of real numbers $a_k, k \in N$ and every pair of non-negative integers n, k, set

$$\Delta^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j}$$

The sequence $(a_k, k \in N)$ is called completely monotone if $\Delta^n a_k \geq 0$ for all integers $n, k \geq 0$. Hausdorff [3] has shown that for a sequence (a_k) of real numbers to be the moment sequence of some non-decreasing g (this case being of particular interest), it is necessary and sufficient that (a_k) be completely monotone. So, completely monotone sequence gives a non-decreasing solution of the moment problem.

In this paper we will show that the results permit a generalization to the situation where (a_k) is a completely monotone sequence of elements of an ordered vector space the definition being the same. This leads to a generalization of the representation theorem for positive linear operators on the space $C(I)$ of continuous real functions on I. Schaefer [4] has considered a completely monotone sequence with values in an ordered locally convex vector space V satisfying some conditions. The results obtained are similar but distinct – neither contains the other, because there need not exist a Hausdorff vector topology on V for which each upper bounded monotone increasing sequence converges in the topology to its supremum [6]. In [2] there is given a simple example of a boundedly complete vector lattice exhibiting this pathology.

2. Assumptions and preliminaries

We consider a (conditionally) σ-complete, weakly σ-distributive vector lattice V satisfying the following two conditions:

(i) Any interval $[a,b] \subset V$ is sequentially o-compact, i.e. for every sequence $(a_i) \subset$
$[a,b]$ there is an o-convergent subsequence (a_{n_i}) (o-means order).

(ii) Every chain in V is at most countable.

We shall need the following notations and simple results (valid for an arbitrary l-group). Let $f : [0,1] \to V$ be a function. We define, as usually, $V(D,f) = \sum_i |f(b_i) - f(a_i)|$, $P(D,f) = \sum_i (f(b_i) - f(a_i))^+$, $N(D,f) = \sum_i (f(b_i) - f(a_i))^-$, $V(f) = V(f,[a,b]) = \sup\{V(D,f);\quad \text{all divisions } D \text{ of } [a,b]\}$, similarly $P(f) = \ldots$, $N(f) = \ldots$. We have $V(D,f) = P(D,f) + N(D,f)$.

Lemma 1. *V(f)=P(f)+N(f).*

Proof. $N(f) \geq N(D,f)$, $P(f) \geq P(D,f)$ for every D. It follows $U(f) + P(f) \geq N(D,f) + P(D,f) = V(D,f)$ for every D. It follows that $N(f) + P(f) \geq V(f)$.

For any decompositions D_1 and D_2 there exists a common refinement D. Then we have

$$N(D_1,f) + P(D_2,f) \leq N(D,f) + P(D,f) = V(D,f) \leq V(f)$$

Fix D_2. Then

$$N(f) = \sup\{N(D_1,f); D_1\} \leq V(f) - P(D_2,f)$$

$$P(D_2,f) \leq V(f) - N(f) \quad \text{for all} \quad D_2$$

It follows $P(f) \leq V(f) - N(f)$ which implies $N(f) + P(f) \leq V(f)$. □

Theorem 1. *The function f has o-bounded variation if and only if f is a difference of two non-decreasing functions, namely, $f(x) = (p(x) + f(x)) - n(x)$, where $p(x) = P(f,[0,x])$, $n(x) = N(f,[0,x])$.*

Proof. If $v(x) = V(f,[0,x])$, then $v(x) = p(x) + n(x)$ by Lemma 1. For a particular D, $P(D,f) - N(D,f) = f(b) - f(a)$, hence

$$P(f) = f(b) - f(a) + N(f)$$

$$p(x) = f(x) - f(0) + n(x)$$

$$f(x) = (p(x) + f(0)) - n(x). \ \square$$

3. The second Helly-Bray theorem

For the following theorem it suffices the second condition to be satisfied and a V to be σ-complete weakly σ-distributive l-group (lattice ordered group).

Theorem 2. *To every sequence* (f_n) *of non-decreasing functions* $f_n : [0,1] \to [a,b] \subset V$ *there are a non-decreasing function* f *and a subsequence* (f_{n_i}) *of* (f_n) *such that* $f_{n_i}(x) \to f(x)$*, whenever* f *is "continuous" at* x*, i.e.* $\sup\{f(y); y < x\} = \inf\{f(z); z > x\}$.

Proof. Let D be a dense subset of $[0,1]$, $D = \{x_1, x_2, \ldots\}$, let (f_n) be a sequence of non-decreasing functions. The condition (ii) implies that there exists a subsequence (f_n^1) of (f_n) such that $(f_n^1(x_1))$ converges. Further, there exists a subsequence (f_n^2) of (f_n^1) such that $(f_n^2(x_2))$ converges, etc.

Consider (f_n^n). It is a subsequence of (f_n), and $(f_n^n(x_k))$ converges for every k.

Denote $f_i^i = f_{n_i}$, $f_0(x_k) = o-\lim_{i\to\infty} f_{n_i}(x_k)$ and define $f : [0,1] \to [a,b]$ by

$$f(x) = \sup\{f_0(y); y \in D, y \leq x\}$$

Evidently, f is non-decreasing and $f(x) = f_0(x)$ for $x \in D$.

Let f be continuous at x. Then there exists a sequence $(y_k) \subset D$ such that $y_k < x$, $f(y_k) \nearrow f(x)$ and a sequence $(z_k) \subset D$ such that $z_k > x$, $f(z_k) \searrow f(x)$. Since in weakly σ-distributive l-groups o-convergence is equivalent to the D-convergence, there exist $a_{ij} \downarrow 0$, $b_{ij} \downarrow 0$ $(j \to \infty)$ such that for every $\varphi \in N^N$ there exists k_0 such that for every $k \geq k_0$

$$f(x) - f(y_k) = |(f(x) - f(y_k)| < \bigvee_i a_{i\varphi(i)}$$

$$f(z_k) - f(x) = |f(x) - f(z_k)| < \bigvee_i b_{i\varphi(i)}$$

There exist $c_{ij} \downarrow 0$ such that for every φ we have $\bigvee a_{i\varphi(i)} + \bigvee b_{i\varphi(i)} < \bigvee c_{i\varphi(i)}$. Hence we have

$$\limsup_i f_{n_i}(x) \leq \limsup_i f_{n_i}(z_k) = f(z_k) <$$

$$< f(x) + \bigvee_i b_{i\varphi(i)} < f(y_k) + \bigvee c_{i\varphi(i)} =$$

$$= \liminf_i f_{n_i}(y_k) + \bigvee c_{i\varphi(i)} \leq \liminf_i f_{n_i}(x) + \bigvee c_{i\varphi(i)}$$

Hence for every $\varphi \in N^N$ we have

$$\limsup_i f_{n_i}(x) - \liminf_i f_{n_i}(x) < \bigvee c_{i\varphi(i)}$$

and the weak σ-distributivity implies

$$\limsup_i f_{n_i}(x) - \liminf_i f_{n_i}(x) \leq 0. \ \square$$

4. The first Helly Bray theorem

Theorem 3. *Let (g_n) be a sequence of non-decreasing functioms $g_n : [0,1] \to [a,b] \subset V$, $g_n(x) \to g(x)$ for every point of "continuity" of g, g being non-decreasing. Let $h :< 0,1 >\to R$ be a continuous function. Then*

$$\int_{[a,b]} h dg_n \to \int_{[a,b]} h dg$$

Proof. Since points of continuity of g form a dense subset in $[0,1]$ for every integer $k > 1$ there exist points $x_0^{(k)}, x_1^{(k)}, \ldots, x_k^{(k)}$ such that $0 = x_0^{(k)} < x_1^{(k)} < \ldots < x_k^{(k)} = 1$, $x_i^{(k)} - x_{i-1}^{(k)} < \frac{1}{k-1}$.

Put $h_k(0) = h(0), h_k(1) = h(1)$, and $h_k(x) = h(x_{i-1}^{(k)})$ if $x \in [x_{i-1}^{(k)}, x_i^{(k)}]$. Then h_k is a simple function, $h_k \to h$ uniformly and

$$\int h_k dg_n = \sum h_k(x_{i-1}^{(k)})(g_n(x_i^{(k)}) - g_n(x_{i-1}^{(k)})) \to$$

$$\sum h_k(x_{i-1}^{(k)})(g(x_i^{(k)}) - g(x_{i-1}^{(k)})) = \int h_k dg, n \to \infty$$

Uniform convergence of h_k to h implies that for every n there exists k_0 such that for all $k > k_0$ and for all $x \in [0,1]$ we have $|h_k(x) - h(x)| < 1/n$ and there exist $C \in V$ and $(a_n) \in V$, $a_n \downarrow 0$ such that

$$|\int h dg_n - \int h dg| \leq |\int h dg_n - \int h_k dg_n| +$$

$$+|\int h_k dg_n - \int h_k dg| + |\int h_k dg - \int h dg| <$$
$$< \frac{1}{n}(g_n(1) - g_n(0)) + |\int h_k dg_n - \int h_k dg| + \frac{1}{n}(g(1) - g(0)) <$$
$$< \frac{1}{n}C + |\int h_k dg_n - \int h_k dg| < \frac{1}{n}C + a_k$$

for fixed k. Therefore

$$\int h dg_n \to \int h dg. \ \square$$

5. Moment theorem

Now we can prove our moment theorem.

Theorem 4. *A sequence $(a_k) \subset V$ is the moment sequence of a non-decreasing function g on I into V if and only if (a_k) is completely monotone.*

Proof. Let $(a_k, k \in N)$ be completely monotone. For each positive integer n define a step function g_n with jumps at $\frac{m}{n}$ for $m = 0, 1, \ldots, n-1$ by the following process. Let

$$a(j,n) := \binom{n}{j} (-1)^{n-j} \Delta^{n-j} a_j$$

for $j = 0, 1, \ldots, n-1$. Set $g_n(0) = 0, g_n(1) = a_0$, and

$$g_n(x) := \sum_{j=0}^{m-1} a(j,n) \quad (\frac{m-1}{n} < x < \frac{m}{n}).$$

Extend g_n to [0,1] by averaging g_n at all jumps.

For each polynomial $P(x) = \sum_{j=0}^{n} c_j x^j$ put

$$\Lambda(P) = \sum_{j=0}^{n} c_j a_j.$$

Consider the Berstein polynomials

$$B(k,n)(x) := \sum_{j=0}^{n} \binom{n}{j} (\frac{j}{n})^k x^k (1-x)^{n-j},$$

and observe that

$$\Lambda(B(k,n)) = \int_0^1 t^k dg_n(t) \tag{1}$$

for $n, k \in N$.

Since $(a_k, k \in N)$ is completely monotone, it is clear that

$$\sum_{j=0}^{n} |a(j,n)| = a_0.$$

Hence the functions $g_0, g_1, \ldots$ are uniformly of bounded variation on $[0,1]$ with variation a_0. Each function g_n is non-decreasing. Therefore, by the second Helly theorem (Theorem 2) there is a non-decreasing function g such that $g_{n_i}(x) \to g(x), i \to \infty$ for x belonging to a dense subset of $[0,1]$. Then by the first Helly theorem (Theorem 3)

$$\lim_{j\to\infty} \int_0^1 t^k dg_{n_j}(t) = \int_0^1 t^k dg(t)$$

Therefore, by the formula (1) it suffices to show that

$$\lim_{n\to\infty} \Lambda(B(k,n)) = a_k$$

for $k \in N$. For completeness we repeat the argument similar as in the numerical case.

Since $a_0 := \Lambda(B(0,n))$, we may suppose that $k > 0$. A direct calculation verifies that

$$a_k = \sum_{j=k}^{n} \frac{j(j-1)\ldots(j-k+1)}{n(n-1)\ldots(n-k+1)} a(j,n)$$

Indeed, $a_k = \Lambda(x^k)$ and by the binomial theorem we can write

$$x^k = x^k((1-x)+x)^{n-k} = \sum_{j=k}^{n} \frac{j(j-1)\ldots(j-k+1)}{n(n-1)\ldots(n-k+1)} \binom{n}{j} x^j(1-x)^{n-j}$$

Consequently, the definition of Λ implies

$$a_k - \Lambda(B(k,n)) = \sum_{j=k}^{n} \left(\frac{ny(ny-1)\ldots(ny-k+1)}{n(n-1)\ldots(n-k+1)} - y^k\right) a(j,n) - \sum_{j=0}^{k-1} y^k a(j,n) \tag{2}$$

for $y = j/n$. Since $(nx - i)/(n - i)$ converges uniformly to x on $[0,1]$, as $n \to \infty$, it is clear that

$$\lim_{n\to\infty} \Pi_{i=0}^{k-1} \frac{nx - i}{n - i} = x^k$$

uniformly on $[0,1]$. Hence given $\epsilon > 0$, there is an $n_0 > 0$ such that

$$\left|\frac{ny(ny-1)\dots(ny-k+1)}{n(n-1)\dots(n-k+1)} - y^k\right| < \epsilon$$

for $n > n_0$, $y = j/n$, and $k \le j \le n$. Moreover, we can choose n_0 so large that

$$\left|\sum_{j=0}^{k-1} y^k a(j,n)\right| < \left(\frac{k}{n}\right)^k a_0 < \epsilon a_0$$

for $n > n_0$. Therefore, it follows from (2) that

$$|a_k - \Lambda(B(k,n))| < \epsilon(a_0 + a_0)$$

for $n > n_0$. □

6. Integral representation theorem

It is well known that every positive linear form on the space of $C(I)$ is presentable in the form

$$f \to \int_0^1 f(t)dq(t) \tag{3}$$

where q is a non-decreasing function. From the preceeding theorem we obtain the following result.

Theorem 5. *Every positive linear operator L on $C(I)$ into V is presentable in the form*

$$L(f) = \int_0^1 f(t)dq(t) \tag{4}$$

where q is a non-decreasing function on I into V. Conversely, every mapping of the form (4) *is positive.*

Thus (4) represents positive linear operators on $C(I)$ with values in V.

Proof. Put

$$L(t^n) = a_n, \qquad n = 0, 1, \ldots$$

Take

$$\Delta^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} L(t^{k+j}) =$$

$$= L(\sum_{j=0}^{n} (-1)^j \binom{n}{j} t^{k+j}) = L(t^k(1-t)^n) \geq 0$$

since L is positive. Hence, by preceeding theorem

$$L(t^n) = \int_0^1 t^n dq(t)$$

for some non-decreasing function q on I with values in V. By Weierstrass theorem we can extend the last equality for every continuous function on I. □

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