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THE ADJOINT THEOREM ON A-SPACES

Endre Pap[1]
Institute of Mathematics, Faculty of Science, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
Charles Swartz
Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, U.S.A.

Abstract

Recently, E. Pap showed that the adjoint operator T' for any linear operator T with the domain being a normed K-space is bounded. E. Pap and C. Swartz proved a locally convex version of this Adjoint Theorem. In this paper a generalization is given of the Adjoint Theorem on operators with the domain being a locally convex A-space, which was introduced by R. Li and C. Swartz. The obtained results are applied to derive a version of the Closed Graph Theorem. Some limitations for further generalizations of the Closed Graph Theorem and Banach-Steinhaus Theorem with respect to infrabarrelledness are pointed out.

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1. Introduction

In [3], a new wide class of spaces, called A-spaces, was introduced for which a general version of the Uniform Boundedness Principle holds. It is very

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interesting that an A-space does not need to be complete or barrelled, properties which are common assumptions in some of the fundamental theorems of the functional analysis.

We shall prove in this paper a generalization of the Adjoint Theorem from [5]. We have a normed space version of the Adjoint Theorem in a different way than in [5]. We have the normed space version of the Adjoint Theorem 1, [4], that the adjoint operator T' for any linear space is bounded. We generalize this theorem on operators with domains which are locally convex A-spaces. We apply the Adjoint Theorem to obtain a version of the Closed Graph Theorem. As a consequence we obtain that every infrabarrelled A-space is barralled. At the end we shall prove that we cannot drop the assumption of infrabarrelledness in the Closed Graph Theorem and in the Banach-Steinhaus Theorem.

2. The Adjoint Theorem

If τ is the vector topology of a vector space X, a sequence $\{x_n\}$ from X is said to be τ-K convergent if every subsequence of $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is τ-convergent to an element $x \in X$.

A subset $B \subseteq X$ is said to be τ-K bounded if for every sequence $\{x_n\}$ from B and every scalar sequence $\{t_n\}$ such that $t_n \to 0$, the sequence $\{t_n x_n\}$ is τ-K convergent ([1], section 3).

Definition 1. *([3]). A topological vector space (X, τ) is said to be an A-space if every τ-bounded subset of X is τ-K bounded.*

There are a large number of important A-spaces, many of which are not complete or K-spaces (see Proposition 5, Corollary 6, Corollary 7 and Corollary 8 in [3]). For example, if X is a B-space, then $(X, \sigma(X, X'))$ is an A-space which is not barrelled, and not even infrabarrelled (see Theorem 5).

Let X and Y be two locally convex Hausdorff topological vector spaces and X' and Y' the corresponding dual spaces, respectively. Let $T : X \to Y$ be a linear operator. The domain of the adjoint operator, T', is defined to be

$$D(T') = \{y' \in Y' : y'T \in X'\}$$

and $T' : D(T') \to X'$ is defined by $T'y' = y'T$.

We have the following generalization of Pap's adjoint theorem.

Theorem 1. *$T' : D(T') \to X'$ carries $\sigma(Y',Y)$ bounded subsets of $D(T')$ to subsets of X' which are uniformly bounded on $\sigma(X,X')-K$ bounded subsets of X.*

Proof. Let $B \subseteq D(T')$ be $\sigma(Y',Y)$ bounded and $A \subseteq X$ be $\sigma(X,X')-K$ bounded. It suffices to show that if $\{y'_k\} \subseteq B, \{x_k\} \subseteq A$, then $\{< T'y'_k, x_k >\}$ is bounded. For this let $t_k > 0, t_k \to 0$.

Consider the matrix $[< \sqrt{t_i}T'y'_i, \sqrt{t_j}x_j >] = M$. We show that M satisfies conditions (I) and (II) of the Basic Matrix Theorem 2.2 of [1]. Since $\{y'_i\}$ is $\sigma(Y',Y)$ bounded, $\{\sqrt{t_i}y'_i\}$ is $\sigma(Y',Y)$ convergent to 0, so for each j we have $< \sqrt{t_j}y'_i, T\sqrt{t_j}x_j >=< \sqrt{t_j}T'y_i, \sqrt{t_j}x_j >\to 0$ as $i \to \infty$, i.e. the columns of M converge to 0 and condition (I) holds.

For condition (II) let $\{m_j\}$ be any increasing sequence of positive integers. There is a subsequence $\{n_j\}$ of $\{m_j\}$ such that the series $\sum_{j=1}^{\infty} \sqrt{t_{n_j}}x_{n_j}$ is $\sigma(X,X')$ convergent to an element $x \in X$. Therefore,

$$\sum_{j=1}^{\infty} < \sqrt{t_i}T'y'_i, \sqrt{t_{n_j}}x_{n_j} >=< \sqrt{t_i}T'y'_i, x >=< \sqrt{t_i}y'_i, Tx) \to 0$$

and (II) holds.

By the Basic Matrix Theorem 2.2 of [1], $< t_iT'y'_i, x_i >\to 0$ and $\{< T'y'_i, x_i >\}$ is bounded.

Theorem 2. *Let (X,τ) be an A-space for some topology τ which is compatible with respect to the duality between X and X', i.e. $\sigma(X,X') \subseteq \tau \subseteq \tau(X,X')$. Then T' carries $\sigma(Y',Y)$ bounded subsets of $D(T')$ to strongly bounded subsets of X'.*

Proof. Let B be a $\sigma(X,X')$ bounded subset of X. Then it is also τ-bounded, and since (X,τ) is an A-space, it is τ-K bounded. Hence, B is $\sigma(X,X')$ - K bounded. By Theorem 1 the operator T' carries $\sigma(Y',Y)$ bounded subsets of $D(T')$ to subsets of X' which are uniformly bounded on $\sigma(X,X')$ - K bounded subsets of X, hence, on $\sigma(X,X')$ bounded sets.

There exist A-spaces which are not K-spaces (a topological vector space (X,τ) is a K-space if every sequence which converges to 0 is τ-K convergent). Namely, (l^p, weak), $s < p < \infty$, is an A-space which is not a K-space.

If X is a normed A-space, then by Proposition 9 from [3] it is also a K-space and therefore Theorem 2 reduces in this case to the adjoint theorem from [4] and Theorem 11 in [1], section 3. By this normed Adjoint Theorem T' carries norm bounded subsets of $D(T')$ into norm bounded subsets of X', i.e., T' is a bounded linear operator.

If we let $K(X, X')$ be the topology of uniform convergence on $\sigma(X', X)$ - K bounded subsets of X', then we have by Proposition 1 from [7] that $K(X, X')$ is stronger than the Mackey topology $\tau(X, X')$ (can be strictly stronger, Example 5 in [7] but still has the same bounded sets as the Mackey topology-Theorem 3 in [7]). The topology $K(X, X')$ can be strictly weaker than the strong topology (Example 6 in [7]).

3. Closed Graph Theorem

Let $\beta(Y, Y')$ be the strong topology on Y and let $\beta^*(Y, Y')$ be the topology on Y of uniform convergence on $\beta(Y', Y)$ bounded subsets of Y'.

By Theorem 2, in an analogous way as in [5], we have

Theorem 3. *Let (X, τ) be an A-space for some compatible topology τ, i.e. $\sigma(X, X') \subseteq \tau \subseteq \tau(X, X')$. If $D(T') = Y'$, then T is continuous with respect to $\beta^*(X, X')$ and $\beta(Y, Y')$.*

A locally convex space Y is an infra-Pták space (B_r-complete) if every $\sigma(Y', Y)$ dense subspace $D \subseteq Y'$ which is such that $D \cap U^\circ$, where U° is the polar of U in Y', is $\sigma(Y', Y)$ closed for every neighbourhood U of 0 in Y, is $\sigma(Y', Y)$ closed ([3],34.3).

By Theorems 2 and 3 and Theorem 4 from [5] if X is infrabarrelled and Y an infra-Pták space, T is closed, and T' carries equicontinuous subsets of $D(T')$ to strongly bounded subsets of X', then $D(T') = Y'$ there follows

Theorem 4. *Let X be an infrabarrelled A-space for some compatible topology τ. Let Y be an infra-Pták space. If T is closed then T is continuous with respect to the original topology of X, $\beta^*(X, X')$ and $\beta(Y, Y')$.*

In [6], we have proved in an easy way a Closed Graph Theorem without the assumption that X is an A-space.

The preceding theorem and Mahowald's theorem ([2], 34.7 (1)) imply

Theorem 5. *Every infrabarrelled A-space is barrelled.*

This is Corollary 12 from [3], proved in a different way.

4. Some limitations for further generalizations

It is easy to see that we cannot drop in Theorem 4 the assumption of infrabarrelledness. Namely, if we suppose that Theorem 4 holds for any A-space X, then, by Mahowald's theorem, X would have to be a barrelled space. But (X, weak) is an A-space which is not barrelled.

In [3], a Uniform Boundedness Theorem is proved when the domain space is any topological vector A-space. But for its relative - the Banach-Steinhaus theorem, the situation is not the same. Namely, as was pointed out in [3], section 4, the infrabarrelledness assumption is in some sense necessary, but then by Theorem 5 this A-space is barrelled. For infrabarrelled spaces there is the following version of the Banach-Steinhaus theorem (see[2], 39.5 Remark 2):

(BS) Let X be infrabarrelled and Y locally convex. Let $A_i, i \in I$, be a net in $L(X,Y)$ such that for every $x \in X$ the net A_i is τ_b-bounded in $L(X,Y)$ and A_iX converges to an element $A_0x \in Y$. Then $A_0 \in L(X,Y)$ and the convergence of A_i to A_0 is uniform on every precompact set in X.

We shall prove that even a version of (BS), when X is an A-space (without infrabarrelledness) is not true. Namely, if we suppose that (BS) is true when X is an arbitrary A-space, then using the same proof as the proof of Theorem 6 from [5] we would obtain: Let X be an A-space and Y be an infra-Pták space. If T is closed and T' carries equicontinuous subsets of $D(T')$ to strongly bounded subsets of X', then $D(T') = Y'$. Hence by Theorems 2 and 3 we would obtain the Closed Graph Theorem when X is any A-space, which is impossible by the remarks at the beginning of this section.

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