NOVI SAD J. MATH.
VOL. 26, NO. 1, 1996, 77–91

COLOMBEAU'S GENERALIZED FUNCTIONS ON A MANIFOLD

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Abstract

A new approach to the space of Colombeau's generalized functions on a manifold is given.

AMS Mathematics Subject Classification (1991): 46Fxx, 35A27
Key words and phrases: distributions, Colombeau's generalized functions, sheaves, pull back

1. Introduction

There are several approaches to Colombeau's generalized functions on a C^∞–manifold (cf. [1], [5], [10], [11]). It is noted in [5] that the definitions in [1] and [11] do not admit the embedding of distributions and distribution densities (in the sense of [8, Ch 6]) into a space of Colombeau's generalized functions on a manifold which is invariant under diffeomorphisms. Because of that, instead of classes $\mathcal{A}_q$, which are usually used in the definition of Colombeau's functions (or their simplified versions), they introduced in [5] classes which we denote here by $\mathcal{A}_q^\varepsilon$, $q \in \mathbf{N}_0$. Then, they proved that for a diffeomorphism $\mu : \Omega_1 \to \Omega_2$ and $G \in \mathcal{G}(\Omega_2)$ the pull back is well defined. The essence of the problem is shown in the following example:

Let f be a compactly supported continuous function on Ω_2 and φ_ε be an appropriate delta net determined by an element of $\mathcal{A}_q^\varepsilon$. Then $f(\mu(x)) * \varphi_\varepsilon$ and $(f * \varphi_\varepsilon)(\mu(x))$ define the same element of $\mathcal{G}(\Omega_1)$. In the setting of [1] and [11] this is true only in the associated sense.

In the present paper we define Colombeau's space by introducing the spaces $\tilde{\mathcal{E}}_M(\Omega)$, $\tilde{\mathcal{N}}(\Omega)$ and then, $\mathcal{E}_M(\Omega)$, $\mathcal{N}(\Omega)$. This procedure is somehow different from the one given in [5]. We define the sheaf of Colombeau's generalized functions on a manifold X by pulling back the sheaf of Colombeau's generalized functions $\mathcal{G}(\Omega)$, $\Omega \subset \mathbf{R}^n$. Our investigations of Colombeau's generalized functions on a manifold are based on Theorem 1. Although this assertion contains more facts than we need in the sequel, it is important for our approach to this matter. In the last part of Section 4 we give the definition of $\mathcal{G}^H(X)$ and explain the motivation for this. In fact this is the definition of Colombeau's generalized function on a manifold given in [5].

The microlocal properties of Colombeau's generalized functions on a manifold have been investigated in [6].

2. Colombeau's–Meril approach

We recall basic notation and notions of Colombeau's theory (cf. [3] and [9]) but in the sense of a new approach which is given in [5].

Let $(\phi^\varepsilon)_{\varepsilon<\varepsilon_\phi}$, $\varepsilon_\phi > 0$ be a net of smooth functions defined on open sets $\Omega_\varepsilon \ni 0$, $\varepsilon < \varepsilon_\phi$, which are contained in the closed ball with the center at 0 and radius 1 ($B(0,1)$), such that for every compact neighborhood of zero K ($\overset{\circ}{K} \ni 0$) there exists $\varepsilon_K > 0$ such that

$(\phi^\varepsilon)_{\varepsilon<\varepsilon_K}$ is a bounded family in $\mathcal{D}_K$ and

$$\int_{\mathbf{R}^n} \phi^\varepsilon(\lambda)\,\mathrm{d}\lambda = 1, \quad \varepsilon < \varepsilon_K. \tag{1}$$

We denote the set of such nets by $\mathcal{A}_0^\varepsilon$ and by $\mathcal{A}_q^\varepsilon$, $q \in \mathbf{N}$ the sets of $(\phi^\varepsilon) \in \mathcal{A}_0^\varepsilon$ which satisfy

$$\int_{\mathbf{R}^n} \lambda^\alpha \phi^\varepsilon(\lambda)\,\mathrm{d}\lambda = \mathcal{O}(\varepsilon^q), \quad 1 \le |\alpha| \le q,\ \varepsilon \to 0.$$

Clearly, $\mathcal{A}_0^\varepsilon \supset \mathcal{A}_1^\varepsilon \supset \ldots$.

Let us note, if $\phi^\varepsilon = \phi$, $\varepsilon \in (0,1)$ and ϕ is defined on $\mathbf{R}^n$ such that (1) and $\int_{\mathbf{R}^n} \lambda^\alpha \phi(\lambda)\, d\lambda = 0,\ \ 1 \leq |\alpha| \leq q$, hold, then $\mathcal{A}_q^\varepsilon = \mathcal{A}_q$, $q \in \mathbf{N}_0$, where $\mathcal{A}_q$ is defined in [3] (see also [9]).

If $(\phi^\varepsilon) \in \mathcal{A}_0$, then we put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi^\varepsilon(\frac{x}{\varepsilon})$, $\varepsilon \in (0, \varepsilon_\phi)$, $x \in \Omega_\varepsilon$.

Let Ω be an open set of $\mathbf{R}^n$. Denote by $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}_0(\mathbf{C})$, resp. $\mathcal{E}_0(\mathbf{R})$) a space of nets of complex valued functions on Ω (resp. nets of complex numbers, resp. nets of real numbers) which correspond to $(\phi^\varepsilon) \in \mathcal{A}_0^\varepsilon$,

$$R : ((\phi^\varepsilon), x) \mapsto R((\phi^\varepsilon), x),\ \ (\phi^\varepsilon) \in \mathcal{A}_0,\ x \in \Omega$$

(resp. $R : (\phi^\varepsilon) \mapsto R((\phi^\varepsilon))$) such that for every fixed ε_0, $R((\phi^{\varepsilon_0}), \cdot\,) \in C^\infty(\Omega)$ (resp. $R((\phi^{\varepsilon_0})) \in \mathbf{C}$, resp. $R((\phi^{\varepsilon_0})) \in \mathbf{R}$). Then, Colombeau's spaces of moderate and null nets of functions are defined by

$$\begin{aligned}\tilde{\mathcal{E}}_M(\Omega) \;=\; & \{R \in \mathcal{E}(\Omega) |\ \forall K \subset\subset \Omega\ \forall \alpha \in \mathbf{N}_0^n\ \exists N \in \mathbf{N} \\ & \forall (\phi^\varepsilon) \in \mathcal{A}_N\ \sup_{x \in K} |D^\alpha R(\phi_\varepsilon, x)| = \mathcal{O}\left(\varepsilon^{-N}\right), \varepsilon \to 0\},\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{N}}(\Omega) \;=\; & \{R \in \mathcal{E}(\Omega) |\ \forall K \subset\subset \Omega\ \forall \alpha \in \mathbf{N}_0^n\ \exists N \in \mathbf{N} \\ & \exists (\gamma_q) \in \mathbf{N}^{\mathbf{N}}, \gamma_q \to \infty \text{ as } q \to \infty \\ & \forall q > N\ \forall (\phi^\varepsilon) \in \mathcal{A}_q\ \sup_{x \in K} |D^\alpha R(\phi_\varepsilon, x)| = \mathcal{O}\left(\varepsilon^{\gamma_q}\right), \varepsilon \to 0\}.\end{aligned}$$

We shall write $R(\phi^\varepsilon, x)$ instead of $R((\phi^\varepsilon), x)$. If in the previous definitions $R(\phi^\varepsilon, x) = R(\phi^\varepsilon)$ does not depend on x and the estimates hold for $\alpha = 0$, i. e.

$$|R(\phi^\varepsilon)| = \mathcal{O}\left(\varepsilon^{-N}\right), \varepsilon \to 0 \quad (\text{resp. } |R(\phi^\varepsilon)| = \mathcal{O}\left(\varepsilon^{\gamma_q}\right), \varepsilon \to 0),$$

then a space of such nets of complex numbers is denoted by $\tilde{\mathcal{E}}_{0M}(\mathbf{C})$ (resp. $\tilde{\mathcal{N}}_0(\mathbf{C})$). If they are nets of real numbers, we obtain spaces $\tilde{\mathcal{E}}_{0M}(\mathbf{R})$ and $\tilde{\mathcal{N}}_0(\mathbf{R})$.

We introduce a relation in $\tilde{\mathcal{E}}_M(\Omega)$

> $R_1(\phi^\varepsilon, \cdot\,) \sim R_2(\phi^\varepsilon, \cdot\,)$ if for every compact set $K \subset\subset \Omega$ and every $(\phi^\varepsilon) \in \mathcal{A}_0$ there exists $\varepsilon_{K,\phi}$ such that $R_1(\phi_\varepsilon, x) = R_2(\phi_\varepsilon, x),\ \ x \in K, \varepsilon < \varepsilon_{K,\phi}$.

It is the equivalence relation and the corresponding spaces of classes will be denoted by $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$. We use the same notation for their elements as for the elements of $\tilde{\mathcal{E}}_M(\Omega)$ and $\tilde{\mathcal{N}}(\Omega)$, respectively. We define $\mathcal{E}_{0M}(\mathbf{C})$, $\mathcal{E}_{0M}(\mathbf{R})$, $\mathcal{N}_0(\mathbf{C})$ and $\mathcal{N}_0(\mathbf{R})$ in an appropriate way.

Clearly, $\mathcal{E}_M(\Omega)$, $\mathcal{E}_{0M}(\mathbf{C})$ and $\mathcal{E}_{0M}(\mathbf{R})$ are associative subalgebras of $\mathcal{E}(\Omega)$, $\mathcal{E}_0(\mathbf{C})$ and $\mathcal{E}_0(\mathbf{R})$, respectively, and $\mathcal{N}(\Omega)$, $\mathcal{N}_0(\mathbf{C})$ and $\mathcal{N}_0(\mathbf{R})$ are ideals of $\mathcal{E}_M(\Omega)$, $\mathcal{E}_{0M}(\mathbf{C})$ and $\mathcal{E}_{0M}(\mathbf{R})$, respectively. The space of Colombeau's generalized functions on Ω, Colombeau's complex numbers and Colombeau's real numbers are defined by

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega), \quad \overline{\mathbf{C}} = \mathcal{E}_{0M}(\mathbf{C})/\mathcal{N}_0(\mathbf{C}) \supset \overline{\mathbf{R}} = \mathcal{E}_{0M}(\mathbf{R})/\mathcal{N}_0(\mathbf{R}).$$

The classical complex numbers are embedded into $\overline{\mathbf{C}}$ by $\mathbf{C} \ni z \mapsto R(\phi^\varepsilon) = z$, $(\phi^\varepsilon) \in \mathcal{A}_0$.

The spaces $\mathcal{G}(\Omega)$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{R}}$ are the algebras with respect to the pointwise multiplication of representatives. The elements of $\overline{\mathbf{C}} \subset \mathcal{G}(\Omega)$ are called constant generalized functions:

$$\mathcal{A}_0 \times \Omega \ni ((\phi^\varepsilon), x) \mapsto R(\phi^\varepsilon).$$

Moreover, $\mathcal{G}(\Omega)$ is a differential algebra (with the Leibnitz rule) with respect to the differentiation defined by the differentiation of representatives:

$$[R(\phi^\varepsilon, x)]^{(\alpha)} = [R^{(\alpha)}(\phi^\varepsilon, x)], \ \alpha \in \mathbf{N}_0^n.$$

Let $(\phi^\varepsilon) \in \mathcal{A}_0$, $d(\phi_\varepsilon) = \sup\{x |\, \phi_\varepsilon(x) \neq 0\}, \varepsilon \in (0, \varepsilon_\phi)$ be the net of support numbers and

$$\overline{\Omega}_{2d(\phi_\varepsilon)} = \{x \in \Omega;\ \mathrm{dist}(x, \mathbf{R}^n \backslash \Omega) \geq 2d(\phi_\varepsilon)\}, \quad \varepsilon \in (0, \varepsilon_\phi).$$

These sets are closed subsets of Ω. Denote by $\kappa_{\phi_\varepsilon}$ a net of functions in $C^\infty(\Omega)$ which are equal to 1 on $\overline{\Omega}_{2d(\phi_\varepsilon)}$ and $\mathrm{supp}\ \kappa_{\phi_\varepsilon} \subset \overline{\Omega}_{d(\phi_\varepsilon)}$, $\varepsilon \in (0, \varepsilon_\phi)$. If $\overline{\Omega}_{d(\phi_\varepsilon)}$ is empty for some ε, then $\kappa_{\phi_\varepsilon} \equiv 0$.

The embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ is made via the mapping $f \mapsto \mathrm{Cd}\, f$ where $\mathrm{Cd}\, f = [\widetilde{\mathrm{Cd}}\, f(\phi_\varepsilon, x)]$ and

(2) $(\widetilde{\mathrm{Cd}}\, f)(\phi_\varepsilon, x) = (\kappa_{\phi_\varepsilon} f) * \check{\phi}_\varepsilon(x), \ x \in \Omega_\varepsilon, \varepsilon \in (0, \varepsilon_\phi), ((\phi^\varepsilon), x) \in \mathcal{A}_0^\varepsilon \times \Omega$

($\check{\phi}(x) = \phi(-x)$). Note, $C^\infty(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$ i. e.

$$(\mathrm{Cd}\, f_1)(\mathrm{Cd}\, f_2) = \mathrm{Cd}\,(f_1 f_2) = f_1 f_2, \quad f_1, f_2 \in C^\infty(\Omega).$$

We have

$$\mathrm{Cd}\,(f^{(\alpha)}) = (\mathrm{Cd}\, f)^{(\alpha)},\ f \in \mathcal{D}'(\Omega).$$

$$\Omega \mapsto \mathcal{G}(\Omega),\ \ \Omega \subset \mathbf{R}^n,\ \text{(resp. } \mathcal{E}_M(\Omega)\text{, resp. } \mathcal{N}(\Omega)\text{) is a sheaf.}$$

If $f \in \mathcal{D}'(\Omega)$ and $\Omega_1 \subset\subset \Omega$, then $\mathrm{Cd}\,(f|_{\Omega_1}) = (\mathrm{Cd}\, f)|_{\Omega_1}$ in $\mathcal{G}(\Omega_1)$.

The support of $G \in \mathcal{G}(\Omega)$, $\mathrm{supp}_g G$, is defined as the complement of the largest open set $\Omega_1 \subset \Omega$ such that $G|_{\Omega_1} = 0$. If $f \in \mathcal{D}'(\Omega)$, then $\mathrm{supp}\, f = \mathrm{supp}_g \mathrm{Cd}\, f$.

We denote by $\mathcal{G}_c(\Omega)$ a subspace of $\mathcal{G}(\Omega)$ consisting of compactly supported elements.

For applications, the equality in $\mathcal{G}$ is too strong. Because of that Colombeau has introduced the notion of association.

It is said that an element $Z \in \overline{\mathbf{C}}$ admits an associated complex number $z \in \mathbf{C}$, in short Z is associated with z ($Z \approx z$), if Z has a representative $Z(\phi_\varepsilon)$ such that there exists $N \in \mathbf{N}_0$ such that $\lim_{\varepsilon \to 0} Z(\phi_\varepsilon) = z$ for every $(\phi^\varepsilon) \in \mathcal{A}_q$, $q \geq N$.

If K is a compact subset of Ω, then the integral of $G \in \mathcal{G}(\Omega)$ over K, $\int_K G\, \mathrm{d}x$, is defined by the representative

$$(\phi^\varepsilon) \mapsto \int_K G(\phi^\varepsilon, x)\, \mathrm{d}x,\ (\phi^\varepsilon) \in \mathcal{A}_0.$$

We embed $\mathcal{G}_c(\Omega)$ into $\mathcal{G}_c(\mathbf{R}^n)$ by $G \mapsto \varphi G$, where $\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ on $\mathrm{supp}\, G$. (If $G \in \mathcal{G}_c(\Omega)$, we will use in the sequel the notation φ for a function in $C_0^\infty(\Omega)$ which is equal to one on $\mathrm{supp}\, G$.)

Let $G \in \mathcal{G}_c(\Omega)$ and $\mathrm{supp}\ G = K \subset\subset \Omega$. Then, $\int_\Omega G\, \mathrm{d}x$ is defined by $\int_{\mathrm{supp}\,\varphi} \varphi G\, \mathrm{d}x$.

Note, if $\psi \in C_0^\infty(\Omega)$ and $G \in \mathcal{G}(\Omega)$, then $\int_\Omega G\psi\, \mathrm{d}x \in \overline{\mathbf{C}}$.

An element $G \in \mathcal{G}(\Omega)$ is said to admit an $f \in \mathcal{D}'(\Omega)$ as an associated distribution, $G \approx f$, if for every $\psi \in C_0^\infty(\Omega)$ $\int_\Omega G\psi\, \mathrm{d}x$ is associated with $< f, \psi >$ ($\in \mathbf{C}$). If $f \in \mathcal{D}'(\Omega)$, then $\mathrm{Cd}\, f \approx f$.

3. Some notions of sheaf theory

Let X be a topological space and $P = (S_U, r^U_V)$ a presheaf of complex vector spaces over X. Recall a presheaf is a sheaf if the following conditions hold:

(*) Let $U = \bigcup_{\alpha\in I} U_\alpha$ and $f \in S_U$ satisfy $r^U_{U_\alpha}(f) = 0$, $\alpha \in I$. Then, $f = 0$.

(**) Let $f_\alpha \in S_{U_\alpha}$, $\alpha \in I$, $r^{U_\alpha}_{U_\alpha\cap U_\beta}(f_\alpha) = r^{U_\beta}_{U_\alpha\cap U_\beta}(f_\beta)$, $\alpha, \beta \in I$ and let $U = \bigcup_{\alpha\in I} U_\alpha$. Then, there is $f \in S_U$ such that $r^U_{U_\alpha}(f) = f_\alpha$, $\alpha \in I$.

If $f \in S_U$, then $\hat{f}_x$ denotes the corresponding germ $\hat{f}_x \in S_x$ at $x \in U$. If a presheaf is a sheaf, we will use the notation ξ for it, and since the mapping

$$\Phi_U : f \mapsto (\Phi_U(f) : x \mapsto \hat{f}_x,\ x \in U), \tag{3}$$

is an isomorphism $S_U \to \Gamma(U, \xi)$, we will use for P the notation

$$\xi = (\Gamma(U, \xi), i^U_V),$$

where i^U_V is the restriction mapping.

Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X and $U \subset X$ be an open set. We define

$$S^{\mathcal{U}}_U = \left\{ (f_i)_{i\in I} \in \prod_{i\in I} S_{U_i\cap U} \mid r_{U_i\cap U_j\cap U}(f_i) = r_{U_i\cap U_j\cap U}(f_j),\ i, j \in I \right\}, \tag{4}$$

where $r_U(f) = \{\hat{f}_x | x \in U\}$;

$$\Psi^{\mathcal{U}}_U : S_U \to S^{\mathcal{U}}_U \ \text{ with }\ \Psi^{\mathcal{U}}_U(f) = (r^U_{U_i\cap U}(f))_{i\in I}.$$

If $V \subset X$ is also open, we define a homomorphism of vector spaces $r^{\mathcal{U}U}_{\ \ V} : S^{\mathcal{U}}_U \to S^{\mathcal{U}}_V$,

$$r^{\mathcal{U}U}_{\ \ V}((f_i)_{i\in I}) = (r^{U_i\cap U}_{U_i\cap V}(f_i))_{i\in I} \ \text{ for } (f_i)_{i\in I} \in S^{\mathcal{U}}_U.$$

It is easy to check that $P^{\mathcal{U}} = (S^{\mathcal{U}}_U, r^{\mathcal{U}U}_{\ \ V})$ is presheaf and that $\Psi^{\mathcal{U}} : P \to P^{\mathcal{U}}$ is a homomorphism of presheaves of vector spaces.

If $\mathcal{V} = (V_j)_{j\in J}$ is another open cover for X and the mapping $\tau : J \to I$ is such that

$$V_j \subset U_{\tau(j)},\ j \in J, \tag{5}$$

then we may define a homomorphism of presheaves $\tau^{\mathcal{U}}_{\mathcal{V}} : P^{\mathcal{U}} \to P^{\mathcal{V}}$ for which $\Psi^{\mathcal{V}} = \tau^{\mathcal{U}}_{\mathcal{V}} \circ \Psi^{\mathcal{U}}$. It suffices to choose

$$\tau^{\mathcal{U}}_{\mathcal{V}\,U}((f_i)_{i\in I}) = (r^{U_{\tau(j)}\cap U}_{V_j\cap U}(f_{\tau(j)}))_{j\in J} \text{ for } (f_i)_{i\in I} \in S^{\mathcal{U}}_U.$$

The homomorphism $\tau^{\mathcal{U}}_{\mathcal{V}}$ does not depend on the choice of mapping $\tau : J \to I$ for which (5) holds. Then $(P^{\mathcal{U}}, \tau^{\mathcal{U}}_{\mathcal{V}})$ is an inductive spectrum of presheaves of vector spaces over X.

Theorem 1.

(i) *Let P be a presheaf. Then the presheaf* $\operatorname{ind}\lim_{\mathcal{U}} P^{\mathcal{U}}$ *is the associated sheaf for P.*

(ii) *Let $\mathcal{U}$ be an arbitrary covering of X. Then the necessary and sufficient condition for $\Psi^{\mathcal{U}} : P \to P^{\mathcal{U}}$, to be*

- *monomorphism, is that the presheaf P satisfies condition (*);*
- *epimorphism, is that the presheaf P satisfies condition (**).*

(iii) *If P is a sheaf, then P and $P^{\mathcal{U}}$ are isomorphic.*

Proof. We will prove only (i) since (ii) and (iii) are simple. First we shall recall some notions (cf. [7]).

Let D be a commutative diagram of presheaves over X and P be a presheaf over X. A family $(f_R)_{R\in A}$ of homomorphisms of presheaves $f_R : R \to P$, whose indices are all presheaves which appear in D, is called a co–cone for the diagram D with vertex P if the diagram, which is obtained from D by adding the presheaf P and homomorphisms f_R, is commutative.

Homomorphism of a co–cone $(f_R)_{R\in A}$ with vertex P into a co–cone $(g_R)_{R\in A}$ with vertex Q is a homomorphism of presheaves $h : P \to Q$ such that D, P, Q and both families $(f_R)_{R\in A}$ and $(g_R)_{R\in A}$ constitute a commutative diagram.

All the co–cones of a diagram D constitute a category. An object of a category from which starts exactly one arrow (morphism) to every object is called the starting object. It is also called the universal co–cone of the diagram D. A vertex of a universal co–cone of the commutative diagram

D is called the co–limit of D. The co–limit of a commutative diagram is determined up to an isomorphism.

An inductive spectrum is a commutative diagram. The co–limit of an inductive spectrum is its inductive limit. Hence, the co–limit of a commutative diagram is called the inductive limit.

Denote by ξ the associated sheaf for P and by $\tilde{P}$ his presheaf of sections. Note, (3) defines a homomorphism of presheaves $\Phi : P \to \tilde{P}$, where $\Phi_U : S_U \to \Gamma(U,\xi)$ is homomorphism of vector spaces. If $(f_i)_{i\in I} \in S_U^{\mathcal{U}}$ and $x \in U_i \cap U_j \cap U$, then $r_x^{U_i\cap U}(f_i) = r_x^{U_j\cap U}(f_j)$. This enables us to define a homomorphism of presheaves $\Theta^{\mathcal{U}} : P^{\mathcal{U}} \to \tilde{P}$ by $\Theta_U^{\mathcal{U}}((f_i)_{i\in I}) = s$, where the section $s \in \Gamma(U,\xi)$ is defined $s(x) = r_x^{U_i\cap U}(f_i)$ for $x \in U_i \cap U$ and $i \in I$. Let $\tau^{\mathcal{U}}_{\mathcal{V}\,U}((f_i)_{i\in I}) = (f_j)_{j\in J} \in S_U^{\mathcal{V}}$. It is obvious that $\Theta_U^{\mathcal{V}}((f_j)_{j\in J}) = \Theta_U^{\mathcal{U}}((f_i)_{i\in I})$, because the sections in $(f_j)_{j\in J}$ are the restrictions of sections f_i and they have equal germs at all points $x \in U$. So, we have $\Theta^{\mathcal{V}} \circ \tau_{\mathcal{V}}^{\mathcal{U}} = \Theta^{\mathcal{U}}$, i. e. the presheaf $\tilde{P}$ is the vertex of a co–cone of a commutative diagram which consists of the objects $P^{\mathcal{U}}$ and morphisms $\tau_{\mathcal{V}}^{\mathcal{U}}$. We need to prove that the co–cone is universal.

Assume that a presheaf $\approx\!\!P$ is a vertex of a co–cone consisting of homomorphisms $g^{\mathcal{U}} : P^{\mathcal{U}} \to \overset{\approx}{P}$. It is sufficient to decompose $g^{\mathcal{U}}$ into two homomorphisms

$$P^{\mathcal{U}} \xrightarrow{\Theta^{\mathcal{U}}} \tilde{P} \xrightarrow{g} \overset{\approx}{P}$$

in such a way that g does not depend on $\mathcal{U}$ and then to prove the uniqueness of such a g.

Let us show that $\theta^{\mathcal{U}}$ is surjective. Let $s \in \Gamma(U,\xi)$. Then for every $x \in U$ there exists an open set $V_x \subset U$ and $\alpha_x \in S_{V_x}$ such that

$$s_{|V_x}(t) = r_t^{V_x}(\alpha_x).$$

If $V_x \cap V_y \neq \emptyset$ $(x, y \in U)$, then there holds

$$r_{V_x\cap V_y}(\alpha_x) = r_{V_x\cap V_y}(\alpha_y).$$

Now we define a family $\mathcal{W}$ of open sets which cover X as follows. The family $\mathcal{V}$ of open sets V_x, $x \in U$ is a subset of this family. Let $W \in \tau$ and $W \cap U \neq \emptyset$. We denote by $\mathcal{W}_W$ a family of open sets which contains W and W_z, $z \in U$, which satisfy the following conditions:

(i) $\bigcup_{z\in U} W_z = U$,

(ii) $\bigcup_{z\in U\cap W} W_z = U \cap W$,

(iii) $W_z \subset V_z$, for every $z \in U$.

Let H be the interior of $X \backslash U$. Then we put

$$\mathcal{W} = \mathcal{V} \cup \bigcup_{W\cap U\neq\emptyset} \mathcal{W}_W \cup \{H\} = \{O_i | \, i \in I\}.$$

One can prove that

$$\Theta_U^{\mathcal{W}}((f_i)_{i\in I}) = s,$$

where $(f_i)_{i\in I} = \tau_{\mathcal{W}}^{\mathcal{V}}(\alpha_x)_{x\in U}$.

We define a homomorphism $g_U : \Gamma(U,\xi) \to \overset{\approx}{S}_U$ by $g_U(s) = h_U^{\mathcal{U}}((f_i)_{i\in I})$. We need to prove that g_U is well defined and that the right side of the equality does not depend on the choice of the family $(f_i)_{i\in I}$ which determines s. It is obvious that $h_U^{\mathcal{U}}((f_i)_{i\in I}) = h_U^{\mathcal{V}}((f_j)_{j\in J})$, if $(f_j)_{j\in J} = \tau_{\mathcal{V}\,U}^{\mathcal{U}}(f_i)_{i\in I}$. □

Remark 1. *Note* $\Theta^{\mathcal{U}} \circ \Psi^{\mathcal{U}} = \Phi$.

Remark 2. *Instead of (4) we may use the following definition:*

$$S_U^{\mathcal{U}} = \left\{ (f_i)_{i\in I} \in \prod_{i\in I} S_{U_i\cap U} \mid r_{U_i\cap U_j\cap U}^{U_i\cap U}(f_i) = r_{U_i\cap U_j\cap U}^{U_j\cap U}(f_j),\ i,j \in I \right\},$$

and the corresponding definition of $r^{\mathcal{U}}{}_V^U$ *to obtain a presheaf similar to* $P^{\mathcal{U}}$*, but in this case we do not know whether (i) in Theorem 1 is true. However, we still have claims (ii) and (iii) of this theorem which we only need for the following section.*

4. Colombeau's generalized functions on manifolds

Let X be a C^∞–manifold of dimension n and let $\mathcal{F} = \{(X_\kappa, \kappa)\}$ be a maximal family of coordinate systems. We denote by Λ a set of all diffeomorphisms which appear in $\mathcal{F}$. Let

(6) $$\mathcal{F}^1 = \{(X_\kappa, \kappa) | \, \kappa \in \Lambda^1\},\ \ \Lambda^1 \subset \Lambda,$$

be an atlas of X. We denote $\mathcal{U}^1 = \{X_\kappa | \kappa \in \Lambda^1\}$.

Let ξ be a sheaf of complex vector spaces on X. We have shown that the sheaf $\xi^{\mathcal{U}^1}$ is isomorphic to ξ. For every open set $O \subset X$,

$$\Gamma(O,\xi) \text{ is isomorphic to } \Gamma(O,\xi^{\mathcal{U}^1}),$$

and $\Gamma(O,\xi^{\mathcal{U}^1})$ is determined by $\Gamma(X_\kappa,\xi^{\mathcal{U}^1})$, $\kappa \in \Lambda^1$ which are isomorphic with $\Gamma(X_\kappa,\xi)$, $\kappa \in \Lambda^1$. This implies that the vector spaces

$$\Gamma(X_\kappa,\xi), \quad \kappa \in \Lambda^1$$

determine the sheaf ξ, uniquely up to an isomorphism.

The open sets of $\mathbf{R}^n$ are indicated by $\sim$ over the capital letter: $\tilde{U}$, $\tilde{V}$, $\tilde{X}_\kappa$, ...

Theorem 2. *Let ξ_0 be a sheaf of complex vector spaces on $\mathbf{R}^n$ with the property:*

(A) For every diffeomorphism $h : \tilde{U} \to \tilde{U}'$ the mapping

$$h^* : \Gamma(\tilde{U}',\xi_0) \to \Gamma(\tilde{U},\xi_0),$$

given by

$$\Gamma(\tilde{U}',\xi_0) \ni f \to h^*f = f \circ h \in \Gamma(\tilde{U},\xi_0),$$

is an isomorphism.

Then ξ_0 determines a sheaf ξ on X such that $\xi = \xi_{0|\tilde{U}}$ if $X = \tilde{U}$.

Proof. Let $\mathcal{U} = \{X_\kappa | \kappa \in \Lambda\}$ and

$$S_{X_\kappa} = \kappa^*\Gamma(\tilde{X}_\kappa,\xi_0), \text{ where } (X_\kappa,\kappa) \in \mathcal{F}.$$

The family $\{S_{X_\kappa} | \kappa \in \Lambda\}$ is uniquely determined by a family $\kappa^*\Gamma(\tilde{X}_\kappa,\xi_0)$, $\kappa \in \Lambda^1$, for any atlas (6). This follows from the considerations which proceed Theorem 2.

Let us put $\mathcal{U} = \{X_\kappa | \kappa \in \Lambda\}$ and define for every open set $U \subset X$,

$$S_U^{\mathcal{U}} = \left\{ (f_i)_{i\in I} \in \prod_{\kappa\in\Lambda} S_{U_\kappa \cap U} | \, r_{X_\kappa \cap X_{\kappa'} \cap U}(f_\kappa) = r_{X_\kappa \cap X_{\kappa'} \cap U}(f_{\kappa'}) \right\}.$$

Clearly, $U \mapsto S_U^{\mathcal{U}}$ is a sheaf. We denote it by $\xi = \xi^{\mathcal{U}}$. Similarly as above, every atlas of the form (6) and the corresponding covering $\mathcal{U}^1$ determine the sheaf $U \mapsto S_U^{\mathcal{U}^1}$, $U \subset X$. One can easily prove that this sheaf $\xi^{\mathcal{U}^1}$ is equal to ξ.

Let $X = \tilde{U}$. The definition of $\xi_{0|U}$ and (A) imply $\xi = \xi_{0|\tilde{U}}$. □

Theorem 3 (and Definition). *Let ξ_0 be one of sheaves $\tilde{U} \mapsto \mathcal{E}_M(\tilde{U})$ or $\tilde{U} \mapsto \mathcal{N}(\tilde{U})$ or $\tilde{U} \mapsto \mathcal{G}(\tilde{U})$. Then for every one of them (A) holds and they determine the corresponding sheaves on X which we call a sheaf of moderate functions $U \mapsto \mathcal{E}_M^C(U)$, null functions $U \mapsto \mathcal{N}^C(U)$ and Colombeau's generalized functions on a manifold $U \mapsto \mathcal{G}^C(U)$, $U \subset X$.*

The sheaf $\tilde{U} \mapsto \mathcal{N}(\tilde{U})$ is fine (as well as $\tilde{U} \mapsto \mathcal{E}_M(\tilde{U})$). This implies that $H^q(\mathbf{R}^n, \mathcal{N}) = 0$, $q \geq 1$. Thus, $\mathcal{G}(\tilde{U}) = \mathcal{E}_M(\tilde{U})/\mathcal{N}(\tilde{U})$ and $\mathcal{G}^C(U) = \mathcal{E}_M^C(U)/\mathcal{N}^C(U)$. The sheaf property of Colombeau's generalized functions on $\mathbf{R}^n$ is proved directly by checking conditions (*) and (**).

Damsma and de Roever [11] have defined the sheaf of ultrafunctions on a manifold by using a simplified version of Colombeau's theory. They imbed distribution densities into ultrafunctions via de Rham's regularizations. By the use of families $\mathcal{A}_q^\varepsilon$, we can reformulate their definitions in [11] of the sheaf of ultrafunctions which we denote here by $U \mapsto \mathcal{G}^R(U)$, $U \subset X$.

Theorem 4 ([6]). *The sheaves $U \mapsto \mathcal{E}_M^R(U)$, $U \mapsto \mathcal{N}^R(U)$, $U \mapsto \mathcal{G}^R(U)$, $U \subset X$, are isomorphic with the sheaves $U \mapsto \mathcal{E}_M^C(U)$, $U \mapsto \mathcal{N}^C(U)$, $U \mapsto \mathcal{G}^C(U)$, respectively.*

We will use the notation from the beginning of this section. Let (X_κ, κ), $(X_{\kappa'}, \kappa') \in \mathcal{F}$, $\mu = \kappa \circ \kappa'^{-1}$ and $\phi^\varepsilon \in \mathcal{A}_q^\varepsilon$. If for every coordinate system $\kappa : X_\kappa \to \tilde{X}_\kappa \subset \mathbf{R}^n$ (X_κ is open in X) there is a function

$$\mathcal{A}_0^\varepsilon \times X_\kappa \ni (\phi^\varepsilon, x) \mapsto G_\kappa(\phi^\varepsilon, x) \in \mathcal{E}_M(\tilde{X}_\kappa) \;\; (\text{resp. } \in \mathcal{N}(\tilde{X}_\kappa))$$

such that for every $\kappa, \kappa' \in \Lambda$ and every $\tilde{K} \subset\subset \kappa'(X_\kappa \cap X_{\kappa'})$ there exists $\varepsilon_0 > 0$ such that

$$(7) \qquad \mu^* G_\kappa(\phi^\varepsilon, x') = G_{\kappa'}(\phi^\varepsilon, x') = G_\kappa(\phi^\varepsilon, \mu(x')), \;\; \varepsilon < \varepsilon_0, x' \in \tilde{K},$$

we call the system $G = \{G_\kappa\}$ a moderate function on X; $G \in \tilde{\mathcal{E}}_M^H(X)$ (resp. a null function on X; $G \in \tilde{\mathcal{N}}^H(X)$). We use the notation $G_\kappa = G \circ \kappa^{-1}$.

Theorem 5. *The vector spaces $\tilde{\mathcal{N}}^H(X)$ and $\tilde{\mathcal{E}}^H_M(X)$ are closed under multiplication $GR = \{G_\kappa R_\kappa\}$. Moreover, $\tilde{\mathcal{N}}^H(X)$ is an ideal of $\tilde{\mathcal{E}}^H_M(X)$.*

Proof. It is a consequence of similar properties for algebras of Colombeau's functions on an open subset of $\mathbf{R}^n$. □

Definition 1. *The space of Colombeau's generalized functions on X, $\tilde{\mathcal{G}}^H(X)$ is defined by $\tilde{\mathcal{G}}^H(X) = \tilde{\mathcal{E}}^H_M(X)/\tilde{\mathcal{N}}^H(X)$. Their elements are $G = \{G_\kappa\}$, where G_κ is represented by $G_\kappa(\phi^\varepsilon, \cdot) \in \mathcal{E}_M(\tilde{X}_\kappa)$.*

Let U be an open set in X. We define $\tilde{\mathcal{E}}^H_M(U)$, $\tilde{\mathcal{N}}^H(U)$ and $\tilde{\mathcal{G}}^H(U) = \tilde{\mathcal{E}}^H_M(U)/\tilde{\mathcal{N}}^H(U)$ in an appropriate way.

Theorem 6 ([6]).

(i) Let $\mathcal{F}^1$ be an atlas for X. If for every $\kappa \in \Lambda^1$ we have a moderate function $\tilde{G}_\kappa \in \mathcal{E}_M(\tilde{X}_\kappa)$ (resp. $\mathcal{N}(\tilde{X}_\kappa)$) and (7) is valid when κ and κ' belong to $\mathcal{F}^1$, then there exists one and only one moderate function $G \in \tilde{\mathcal{E}}^H_M(X)$ (resp. $G \in \tilde{\mathcal{N}}^H(X)$) such that $G \circ \kappa^{-1} = G_\kappa$ for every $\kappa \in \Lambda^1$.

In particular, $\tilde{\mathcal{G}}^H(X)$ is uniquely determined by an atlas $\mathcal{F}^1$.

(ii) The presheaf $U \mapsto \tilde{\mathcal{G}}^H(U)$, $U \subset X$ is a sheaf.

(iii) The sheaves

$$U \mapsto \mathcal{E}^C_M(U),\ U \mapsto \mathcal{N}^C(U),\ U \mapsto \mathcal{G}^C(U),\ \ U \subset X \tag{8}$$

determine uniquely spaces of moderate, null and generalized functions $\tilde{\mathcal{E}}^H_M(X)$, $\tilde{\mathcal{N}}^H(X)$ and $\tilde{\mathcal{G}}^H(X)$, respectively. Conversely, the spaces $\tilde{\mathcal{E}}^H_M(X)$, $\tilde{\mathcal{N}}^H(X)$ and $\tilde{\mathcal{G}}^H(X)$ in a unique way determine sheaves in (8).

The notions of equality in $(g.d)$ sense and in associated sense are introduced in $\tilde{\mathcal{G}}^H(X)$ in an obvious way.

Let $u = \{u_\kappa\}$ be a distribution on X ($u \in \mathcal{D}'(X)$). Recall, $\mathcal{F} = (X_\kappa, \kappa)$ is a maximal family of coordinate systems and $u_\kappa \in \mathcal{D}'(\tilde{X}_\kappa)$, $\tilde{X}_\kappa = \kappa(X_\kappa)$. Then the embedding $\mathcal{D}'(X) \to \tilde{\mathcal{G}}^H(X)$ is defined by $u \mapsto \mathrm{Cd}\, u = (\mathrm{Cd}\, u_\kappa)$. Clearly, it is determined by any atlas in the sense of Theorem 6 (i).

We investigated in [6] the space $\mathcal{G}^H(X)$ which is accommodated to the definition of distributions on a manifold (cf. [8, Section 6.3]). Here, we present just the motivation for the definition of $\mathcal{G}^H(X)$ (see also [5]).

We will use the same notation as in the previous paragraph for the manifold X and its maximal coordinate system $\mathcal{F} = \{(X_\kappa, \kappa) | \kappa \in \Lambda\}$.

Let U_1 and U_2 be open sets in X, $\mu : U_1 \to U_2$ be a diffeomorphism and $u = (u_\kappa)$ be a distribution on U_2. Then $\mu^* \mathrm{Cd}\, u \approx \mathrm{Cd}\, \mu^* u$ but

$$\mu^* \mathrm{Cd}\, u \neq \mathrm{Cd}\, \mu^* u. \tag{9}$$

This will be proved in [6]. Because of that we define $\mathcal{E}^H(X)$, $\mathcal{N}^H(X)$ and consequently $\mathcal{G}^H(X)$ in a way such that equality holds in (9).

Let $\mu : \Omega_1 \to \Omega_2$ be a C^∞–diffeomorphism and K_n be a sequence of compact sets such that $K_n \subset\subset K_{n+1}$, $\bigcup_{n=1}^{\infty} K_n = \Omega_1$. Let $(\phi^\varepsilon) \in \mathcal{A}_0^\varepsilon$. There exists ε_n such that,

for $x \in K_n$ and $\varepsilon \leq \varepsilon_n$, the function

$$\tilde{\varphi}^\varepsilon(\xi) = \begin{cases} \frac{1}{|J_\mu(\mu^{-1}(\mu(x+\varepsilon\xi)))|} \varphi^\varepsilon\left(\frac{\mu^{-1}(\mu(x)+\varepsilon\xi)-x}{\varepsilon}\right), & |\xi| < 1 \\ 0, & |\xi| \geq 1 \end{cases},$$

is well defined.

The construction implies that $(\tilde{\varphi}^\varepsilon) \in \mathcal{A}_0$. In fact, it belongs to $\mathcal{A}^\varepsilon_{[q/2]}$, where $[q/2]$ is the integer part of $q/2$. It was proved in [5] and used there for the definition of $\mu^\oplus : \mathcal{E}_M(\Omega_2) \to \mathcal{E}_M(\Omega_1)$ (resp. $\mathcal{N}(\Omega_2) \to \mathcal{N}(\Omega_1)$) and of $\mu^\oplus : \mathcal{G}(\Omega_2) \to \mathcal{G}(\Omega_1)$ as follows.

Let $R(\phi^\varepsilon, y) \in \mathcal{E}_M(\Omega_2)$. Then,

$$(\mu^\oplus R)(\phi^\varepsilon, x) = \tilde{R}(\phi^\varepsilon, x) = R(\tilde{\phi}^\varepsilon, \mu(x)).$$

This is an element of $\mathcal{E}_M(\Omega_1)$.

Thus, we define $\mu^\oplus : \mathcal{G}(\Omega_2) \to \mathcal{G}(\Omega_1)$ by $\mu^\oplus G = [\mu^\oplus R]$, where $G = [R]$.

Theorem 7 ([5]).

(i) $\mu^\oplus(\mathcal{N}(\Omega_2)) = \mathcal{N}(\Omega_1)$ *and* $\mu^\oplus(\mathcal{E}_M(\Omega_2)) = \mathcal{E}_M(\Omega_1)$.

(ii) $\mu^{\oplus}(Cdf) = Cd(\mu^{\oplus}f)$, $f \in \mathcal{D}'(\Omega_2)$.

Let (X_κ, κ), $(X_{\kappa'}, \kappa') \in \mathcal{F}$, $\mu = \kappa \circ \kappa'^{-1}$ and $\phi^\varepsilon \in \mathcal{A}_q$. Let $(\tilde{\phi}^\varepsilon)$ be defined as above. A family $G = \{G_\kappa\}$ is called a moderate function on X; $G \in \mathcal{E}_M^H(X)$ (resp. a null function on X; $G \in \mathcal{N}(X)$) if the functions

$$\mathcal{A}_0^\varepsilon \times X_\kappa \ni (\phi^\varepsilon, x) \mapsto G_\kappa(\phi^\varepsilon, x) \in \mathcal{E}_M(\tilde{X}_\kappa) \text{ (resp. } \in \mathcal{N}(\tilde{X}_\kappa)), \quad \kappa \in \Lambda,$$

satisfy the following condition:

For every $\kappa, \kappa' \in \Lambda$ and every $\tilde{K} \subset\subset \kappa'(X_\kappa \cap X_{\kappa'})$ there exists $\varepsilon_0 > 0$ such that

$$\mu^{\oplus} G_\kappa(\tilde{\phi}^\varepsilon, x) = G_{\kappa'}(\phi^\varepsilon, x') = G_\kappa(\phi^\varepsilon, \mu(x')), \ \varepsilon < \varepsilon_0, x' \in \tilde{K}.$$

Definition 2. *The space of Colombeau's generalized functions on X, $\mathcal{G}^H(X)$ is defined by $\mathcal{G}^H(X) = \mathcal{E}_M^H(X)/\mathcal{N}^H(X)$. Their elements are $G = \{G_\kappa\}$, where G_κ is represented by $G_\kappa(\phi, \varepsilon) \in \mathcal{E}_M(\tilde{X}_\kappa)$.*

Let U be an open set in X. We define $\mathcal{E}_M^H(U)$, $\mathcal{N}^H(U)$ and $\mathcal{G}^H(U) = \mathcal{E}_M^H(U)/\mathcal{N}^H(U)$ in an appropriate way.

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Received by the editors April 15, 1995.