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DIFFERENT LEVELS OF WORD PROBLEMS FOR SOME VARIETIES

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Abstract

We obtain some new results on the varieties having locally solvable word problems and undecidable equational theories, which represent a continuation of the results announced in [3].

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We present a study of the varieties with locally solvable word problems whose equational theories are undecidable. One example of a recursively axiomatized variety in a finitary language having the properties listed above is given in [3].

1. A countable chain of varieties with locally solvable word problems and undecidable equational theories

We quote the result here as it will be used in the sequel.

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Theorem 1. *In the language of the type $\langle 2,1,1,0\rangle$, there exists a variety with locally solvable word problem and undecidable equational theory. This variety is axiomatized by the following identities*

$$
(1)\qquad
\begin{array}{rcl}
x\cdot 0 &\approx& 0\\
x^2 &\approx& 0\\
x\cdot y &\approx& y\cdot x\\
x\cdot(y\cdot z) &\approx& (x\cdot y)\cdot z
\end{array}
\qquad
\begin{array}{rcl}
f(0) &\approx& 0\\
f(f(x)) &\approx& f(x)\\
f(x\cdot y) &\approx& 0\\
h(0) &\approx& 0\\
h(h(x)) &\approx& h(x)\\
h(x)\cdot y &\approx& 0\\
f(h(x)) &\approx& h(x)
\end{array}
$$

$$h^k(f(x_1)f(x_2)...f(x_{\varphi(k)}))\approx 0,\ k\in\mathbf{N}$$

where $X=\{\varphi(k): k\in N\}$ is a nonrecursive recursively enumerable set of naturals.

Proof. See [3]. □

One can easily demonstrate that any such variety has globally undecidable word problem, and, furthermore, provides an example of a so-called pseudorecursive variety (in the sense of [7]), hence unifying the previous results of Mekler, Nelson and Shelah [5] and Wells [7].

Even though these properties of a variety might seem rather restrictive, building upon the previous result, it turns out that they do not present a rare "phenomenon". Namely, the following can be proved :

Theorem 2. *In the language of the type $\langle 2,1,1,0\rangle$, there exists an infinite (isomorphic to $\langle\omega;\leq\rangle$) chain of varieties which have locally solvable word problems and undecidable equational theory.*

Proof. Let the variety defined in the statement of the previous theorem be denoted by V_1. Let $\epsilon_n, n\geq 2$ be the identity in $\{\cdot, f, h, 0\}$ of the form :

$$\epsilon_n: f(x_1 f(x_2\ldots f(x_n)\ldots))\approx 0$$

The variety whose set of definitional identities is same as the one for V_1, with the exception of $f(x\cdot y)\approx 0$ being replaced by (ϵ_n), will be denoted by V_n. Obviously,

$$V_1\subsetneq V_2\subsetneq\ldots\subsetneq V_n\subsetneq\ldots$$

Now, we prove that each of these inclusions is strict.

(1) V_1 **is a proper subvariety of** V_2 : Let $\mathcal{C}$ be the algebra with the universe $\{0, a, b, c\}$, operation $h^{\mathcal{C}}$ is identically equal to zero, while the tables for $\cdot^{\mathcal{C}}$ and $f^{\mathcal{C}}$ are given by:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	c	0
b	0	c	0	0
c	0	0	0	0

f	
0	0
a	c
b	c
c	c

Clearly, the semigroup reduct of $\mathcal{C}$ satisfies all definitional identities of V_2 in the language consisting of $\cdot$ solely. It can be easily verified that the other axioms of V_2 are valid in $\mathcal{C}$ as well. Also, note that

$$f(a \cdot b) = f(c) \neq 0,$$

hence, $\mathcal{C} \notin V_1$.

(2) V_n **is a proper subvariety of** V_{n+1}, $n \geq 2$: Let $\mathcal{S}$ be the free semigroup over n free generators $\{a_1, \ldots, a_n\}$ in the semigroup variety defined by $x^2 \approx 0, xy \approx yx, x0 \approx 0x \approx 0$.

The operations $f^{\mathcal{S}}, h^{\mathcal{S}}$ are defined in the following way:

$$f^{\mathcal{S}}(w) = w, \text{ for all } w \in S$$

$$h^{\mathcal{S}}(w) = 0, \text{ for all } w \in S$$

Note that the following identity is valid in $\mathcal{S}$:

$$f(x_1 f(x_2 \ldots f(x_{n+1}) \ldots)) \approx x_1 x_2 \ldots x_{n+1}$$

It is rather obvious that among $x_1, \ldots, x_n$, being assigned any particular values from S, at least two of them must have a letter in common, so the commutative law and $x^2 \approx 0$ yield

$$\mathcal{S} \models f(x_1 f(x_2 \ldots f(x_{n+1}) \ldots)) \approx 0.$$

In order to demonstrate that the identity

(2) $$f(x_1 f(x_2 \ldots f(x_n) \ldots)) \approx 0$$

is not valid in $\mathcal{S}$, substitute $x_1 = a_1, \dots x_n = a_n$. If $\mathcal{S} \models (2)$, we would get

$$a_1 a_2 \dots a_n = 0$$

in $\mathcal{S}$, which, of course, cannot be true.

We quote here the example of the algebra used in the proof of Theorem 1, which served as a main tool in establishing the undecidability of the equational theory of V_1.

Let F be the free semigroup over the countable set of free generators $G = \{a_1, a_2, ..., a_n, ...\}$ in the variety V' given by :

$$\begin{array}{rcl} x \cdot 0 & \approx & 0 \\ x^2 & \approx & 0 \\ x \cdot y & \approx & y \cdot x \\ x \cdot (y \cdot z) & \approx & (x \cdot y) \cdot z \end{array} \tag{3}$$

and let $\infty \notin F$ be an arbitrary element. Denote $F' = F \bigcup \{\infty\}$. Operations $\odot, f$ i h on F' are defined as follows:

$$x \odot y = \begin{cases} x \cdot y & x, y \in F_{V'} \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} x & x \in G \\ 0 & x = 0 \text{ or } x \text{ is a word from } F_{V'} \text{ of length } \geq 2 \\ \infty & \text{otherwise} \end{cases} \tag{4}$$

$$h(x) = \begin{cases} 0 & x = 0 \\ 0 & \text{if} \quad x = b_1 b_2 ... b_t, t \in X, b_i \in G, b_i \neq b_j (i \neq j) \\ \infty & \text{otherwise} \end{cases}$$

where X is a nonrecursive set from Theorem 1.

Using $\mathcal{F}'$, we proved that

$$V_1 \models h(f(x_1) \dots f(x_k)) \text{ iff } k \in X$$

Clearly, $\mathcal{F}' \in V_n$, for $n \geq 2$, which yields (in the essentially same manner as in the proof of Theorem 1) the undecidability of the equational theory of $V_n, n \geq 2$.

The only thing that remains to be shown is that every finitely generated free algebra in $V_n, n \geq 2$ is finite. Let $\mathcal{F}_{m,n}$ denote the free algebra over m free generators $\{a_1, \dots, a_m\}$ in the variety V_n, $n \geq 2$.

Let us call the word over this set of generators of f-depth n, if it is of the form

$$f(w_1 f(w_2 f(w_3 \ldots f(w_n) \ldots)))$$

where $w_1, w_2, \ldots w_n$ are the words in $F_{m,n}$ involving only $\cdot$ and f as functional symbols.

One can easily prove that any word in $\mathcal{F}_{m,n}$ has f-depth at most $n-1$, and, therefore, there are only finitely many nonequivalent (modulo laws of the variety V_n) words of f-depth of at most $n-1$.

Using the induction on the complexity of words containing $\cdot$ and f as the only functional symbols, it is straightforward to check whether any such word, modulo laws of the variety V_n, can be represented as:

$$wt_1 \ldots t_k,$$

where w belongs to the free semigroup over $\{a_1, \ldots, a_m\}$ (w can be the empty word, as well), and $t_1, \ldots t_k$ are the words of f-depth of at most $n-1$.

The finiteness of $\mathcal{F}_{m,n}$ is now a simple consequence of the previously established facts.□

2. Word problem for discriminator varieties

Definition 1. *A ternary discriminator on the set A is the function $t_A : A^3 \mapsto A$, given by:*

$$t_A(a,b,c) = \begin{cases} c, & \text{for } a = b \\ a, & \text{otherwise} \end{cases}$$

Definition 2. *A variety V in the algebraic language $\mathcal{L}$ is said to be a discriminator variety if there exists a term in $\mathcal{L}$ inducing ternary discriminator on the universe of every subdirectly irreducible algebra in V.*

The canonical way of generating discriminator varieties proceeds as follows: Let K be a universally axiomatized class of algebras in language $\mathcal{L}$, and let $\mathcal{L}_t$ denote the language obtained by adding a new ternary functional symbol $t \notin \mathcal{L}$ to $\mathcal{L}$. If $\mathcal{A} \in K$, let $\mathcal{A}^t$ stand for the algebra $\langle \mathcal{A}; t_A \rangle$ in $\mathcal{L}_t$. Let $K^t = \{\mathcal{A}^t : \mathcal{A} \in K\}$. Then, the variety generated by K^t, i. e., $V(K^t)$ is a discriminator variety.

Conversely, every discriminator variety is, up to term equivalence, of the form $V(K^t)$, for some uniquely determined universally axiomatizable class of algebras K.

It can be easily shown that every algebra possessing a discriminator term, i. e., the term inducing the ternary discriminator on it, must be simple and having no nontrivial subalgebras. Hence, for any discriminator variety, the classes of directly irreducible, subdirectly irreducible and simple algebras coincide.

Some other well-known properties of discriminator varieties are listed in the propositions which are to follow.

Theorem 3. *Simple algebras in $V(K)$, where K is a class of algebras endowed with a discriminator term are precisely the members of $SP_U(K_+)$, where K_+ denotes the class K augmented by a one-element algebra (if not already containing one), and $P_U(K)$ denotes the class of all ultraproducts of the members of K.*

Theorem 4. *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be simple algebras in a discriminator variety V. If an algebra $\mathcal{A}$ is a subdirect product of $\mathcal{S}_1, \ldots, \mathcal{S}_n$, then*

$$\mathcal{A} \cong \mathcal{S}_{i_1} \times \ldots \times \mathcal{S}_{i_k},$$

for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

The proofs of these facts can be found in most of the standard universal algebraic texts, like [2].

In [4], the following theorem was proved, giving a uniform procedure of converting the universal sentences into identities on subdirectly irreducible algebras in any discriminator variety.

Theorem 5. *Let V be a discriminator variety in the language $\mathcal{L}$, with the appropriate ternary term $t(x, y, z)$.*

(1) Every finite member of V is isomorphic to a direct product of some subdirectly irreducible algebras from V.

(2) For every universal sentence φ in the language $\mathcal{L}$ of V one can effectively produce the identity ϵ in $\mathcal{L}$, such that

$$\mathcal{A} \models \varphi \Leftrightarrow \epsilon,$$

for every subdirectly irreducible $\mathcal{A} \in V$.

Proof. See [4] □

Corollary 1. *For discriminator varieties, the problem of the decidability of equational theory of V is equivalent to the problem of the decidability of the universal theory of V_{SI}, where V_{SI} denotes the class of subdirectly irreducible members of V. Hence, the decidability of equational theory of V yields the decidability of universal theory of V, and, therefore, the global solvability of the word problem.*

A question which naturally arises is whether the local solvability of the word problem would imply the decidability of the equational theory for a discriminator variety.

Before we proceed to the construction of a counter example which will provide a negative answer to the question raised above, we need to establish an easy fact concerning the recursiveness of a set of formulas. The claim is merely a simple variation of well-known Craig's Theorem which asserts that any recursively enumerably axiomatized theory possesses a recursive set of axioms.

Claim 1. *Every theory axiomatized by a recursively enumerable set of universal sentences has a recursive universal axiomatization.*

Proof. The proof follows immediately from the standard construction used in the proof of Craig's Theorem and the fact that any conjunction of universal sentences is equivalent to a universal sentence. □

Theorem 6. (Willard) *There exists a recursively axiomatized discriminator variety in a finitary language with the locally solvable word problem and undecidable equational theory.*

Proof. Consider the following set of universal sentences in the language consisting of a single binary operation f.

$$(\forall x)(\forall y)(f(x,y) \approx x \vee f(x,y) \approx y) \tag{5}$$

$$(\forall x)(\forall y)(f(x,y) \approx x \Rightarrow f(y,x) \approx y) \tag{6}$$

$$\neg(\exists x_0)(\exists x_1)(\exists x_2)(\exists x_3)(\bigwedge_{i<j} x_i \not\approx x_j \wedge \bigwedge_{i=1}^{3} f(x_0, x_i) \approx x_0) \tag{7}$$

If $G=(V,E)$ is an undirected graph, and if the operation $f^G : V^2 \mapsto V$, is defined by:

$$f^G(x,y) = \begin{cases} x, & \text{if } \{x,y\} \in E \\ y, & \text{otherwise} \end{cases}$$

one can easily show that the algebra $\mathcal{A}_G = \langle G; f^G \rangle$ satisfies the axioms (5) - (7) iff every vertex of G is of a degree at most two.

The converse holds as well, i. e. to any model $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ of these axioms the undirected graph $G_A = (A, E)$ can be assigned, where

$$\{x,y\} \in E \text{ iff } f^{\mathcal{A}}(x,y) = x.$$

Let C_n denote the cyclic graph with n vertices and edges, and let X be a recursively enumerable, nonrecursive set, such that $1 \notin X$. Let $(E_n), n \in X$ stand for the universal sentence

$$(E_n): \neg(\exists x_1)(\exists x_2)\ldots(\exists x_n)[\bigwedge_{i \neq j} x_i \not\approx x_j \wedge \bigwedge_{i=1}^{n-1} f(x_i, x_{i+1}) \approx x_i \wedge f(x_n, x_1) \approx x_1].$$

Intuitively, (E_n) forbids the existence of C_n as a subgraph. According to the described procedure of generating discriminator varieties, we add a new ternary functional symbol t to $\mathcal{L}$, thereby obtaining $\mathcal{L}_t$. Let (T) denote the the universal sentence defining the ternary discriminator

$$(T): \ (\forall x)(\forall y)(\forall z)(x \approx y \Rightarrow t(x,y,z) \approx z \wedge x \not\approx y \Rightarrow t(x,y,z) \approx x).$$

Let K be the class of algebras in $\mathcal{L}_t$ satisfying (5) - (7), (T) and $(E_n), n \in X$. This axiomatization is recursively enumerable, and, by Claim 1, K has a recursive set of universal axioms, which will be denoted by Φ. We have seen (Theorem 5), that there exists a uniform way to convert a universal sentence ψ into an identity $\tau(\psi)$, such that

$$K \models \psi \text{ iff } V(K) \models \tau(\psi).$$

Consider the set of identities $\tau(\Phi) = \{\tau(\phi) : \phi \in \Phi\}$. $\tau(\Phi)$ is an equational basis for $V(K)$, and the thing that is yet to be established is that $\tau(\Phi)$ is indeed recursive. (In terms of the recursion theory, Φ is many-one reducible to $\tau(\Phi)$). First, note that, by the construction presented in the proof of Theorem 5 (see [4]), there are only finitely many universal sentences in $\mathcal{L}_t$, being transformed into any particular identity ϵ in $\mathcal{L}_t$. Moreover, given an arbitrary identity ϵ in $\mathcal{L}_t$, one can effectively recover all those universal sentences (in the prenex normal form) $\varphi_1, \dots, \varphi_k$, such that

$$\tau(\varphi_1) = \dots = \tau(\varphi_k) = \epsilon.$$

Hence, to decide whether the given identity ϵ belongs to $\tau(\Phi)$, effectively construct the appropriate $\varphi_1, \dots, \varphi_k$, and check if some of these belongs to Φ. Therefore, $\tau(\Phi)$ is a recursive set of the equational axioms for $V(K)$.

$V(K)$ is locally finite, thereby having the locally solvable word problem. It remains to be shown that the equational theory of V is undecidable. It suffices to prove, again by Theorem 5, that the universal theory of K is undecidable. Let (E_k), $k \in \omega$ be the universal sentence

$$(E_k): \neg(\exists x_1)(\exists x_2)\dots(\exists x_k)[\bigwedge_{i \neq j} x_i \not\approx x_j \wedge \bigwedge_{i=1}^{k-1} f(x_i, x_{i+1}) \approx x_i \wedge f(x_k, x_1) \approx x_1].$$

We claim :

$$K \models (E_k) \text{ iff } k \in X.$$

The implication $(\Longleftarrow)$ is obvious.

$(\Longrightarrow)$ Let $G = (V, E)$ be a graph, which is a disjoint union of $C_m, m \notin X$. Clearly, the corresponding algebra $\mathcal{A}_G^t$ belongs to K, so, if (E_k) is valid in K, this would imply $k \in X$.

The established equivalence entails the undecidability of the universal theory of K. □

Discriminator varieties are only a part of a wider class of EDPC varieties, arising in the algebraization of different logical systems, and thoroughly studied in [1].

Corollary 2. *There exists a recursively based EDPC variety in a finitary language having locally solvable word problem and undecidable equational theory.*

This result rules out the possibility of obtaining the converse of the following result, due to Blok and Pigozzi:

Theorem 7. (Blok, Pigozzi [1]) *Let V be an EDPC variety having the decidable equational theory. Then the local word problem for V is decidable.*

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