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ON THE CONVERGENCE OF A PARALLEL MULTISPLITTING GENERALIZED ITERATIVE METHOD

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Abstract

We present the Multisplitting Unsymmetric Modified Accelerated Overrelacation (MUSMAOR) we iterative method for solving large nonsingular linear systems of equations $Ax = b$. Besides, we establish some sufficient convergence conditions for this iterative method when the matrix A is an H - matrix.

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1. Introduction

Let us consider the large nonsingular linear system of equations,

$$Ax = b, \tag{1.1}$$

where $A \in R^{n,n}$ and $x, b \in R^n$

In order to solve (1.1) parallel matrix multisplitting iterative methods can be used. They were first proposed by O'Leary and White [1]. and they are based on several splittings of nonsingular matrix, i.e.,

$$A = P_i - Q_i,\ \det\,(P_i) \neq 0,\ i = 1, 2, ..., \alpha \tag{1.2}$$

With these splittings we can construct the following iterative processes for solving (1.1),

$$x^{k+1} = T_i(x^k) = P_i^{-1} Q_i x^k + P_i^{-1} b, \quad i = 1, ..., \alpha \tag{1.3.}$$

and the following parallel multisplitting iterative method,

$$x^{k+1} = \sum_i E_i T_i(x^k),\ \ i = 1, ..., \alpha \tag{1.4}$$

where $E_i \in R^{n,n}$ $(i = 1, 2, ..., \alpha)$ are, the non-negative diagonal matrices verifying $\sum_i E_i = I$, called weighting matrices.

In [2], Frommer and Mayer introduced a relaxation parameter in (1.4), obtaining the Multisplitting method associated to the Successive Overrelaxation method, denoted by MSOR method. They also analyzed its convergence. These results were generalized in [3], where the Multisplitting method associated to the Accelerated Overrelaxation (MAOR) method was proposed, and its convergence analyzed.

In [4] was presented a class of matrix multisplitting multisplitting multiparameter relaxation methods, which includes classes of multisplitting relaxation methods referred above as special cases, as well as the extrapolated multisplitting AOR method [3], and the multisplitting methods associated to the SSOR and SAOR methods.

In [5] was proposed the multisplitting method associated to the generalized iterative method of James [6], and some convergence conditions for it were obtained.

In this paper, we present the multisplitting method associated to the Unsymmetric Modified Accelerated Overrelaxation (MUSMAOR) method, which includes the methods referred above as its special cases, and we also analyze its convergence.

2. Multisplitting algorithms

Let A in (1.1) be a nonsingular real $n \times n$ matrix and $\alpha \in N$. Let us consider

$$A = D - L_i - U_i - W_i, \quad i = 1, 2, ..., \alpha \tag{2.1}$$

where $L_i \in R^{n,n}$ and $U_i \in R^{n,n}$ are strictly the lower and upper triangular matrices, respectively, $W_i \in R^{n,n}$ has zero diagonal entries and D =diag(A) is nonsingular. The splitting (2.1) of matrix A associated with weighting matrices E_i, $(i = 1, 2, ..., \alpha)$ is a multisplitting of the matrix A, and is denoted as $(D - L_i, D - U_i, W_i, E_i)$, $(i = 1, 2, ..., \alpha)$.

We can now introduce the multisplitting method associated to the Unsymmetric Modified AOR (MUSMAOR) method, given by:

$$x^{k+1} = \sum_i E_i F_i(R_1, R_2, \Omega_1, \Omega_2, x^k), \quad k = 0, 1, 2, ... \tag{2.2}$$

where R_j, Ω_j are the diagonal matrices verifying $R_j \geq 0$ and diag $(\Omega_j) > 0$ for $j = 1, 2$ and,

$$F_i(R_1, R_2, \Omega_1, \Omega_2, x^k) = \ell_i(R_1, R_2, \Omega_1, \Omega_2)x^k + b_i(R_1, R_2, \Omega_1, \Omega_2) \tag{2.3}$$

with

$$\begin{aligned} \ell_i(R_1, R_2, \Omega_1, \Omega_2) = {} & (D - R_2U_i)^{-1}[(I - \Omega_2)D+ \\ & (\Omega_2 - R_2)U_i + \Omega_2(L_i + W_i)] \\ & (D - R_1L_i)^{-1}[(I - \Omega_1)D+ \\ & +(\Omega_1 - R_1)L_i + \Omega_1(U_i + W_i)], \ i = 1, ..., \alpha \end{aligned} \tag{2.4}$$

$$\begin{aligned} b_i(R_1, R_2, \Omega_1, \Omega_2) = {} & (D - R_2U_i)^{-1}[(\Omega_1 + \Omega_2 - \Omega_1\Omega_2)D + \Omega_2(\Omega_1 - R_1)L_i \\ & +\Omega_1(\Omega_2 - R_2)U_i + \Omega_1\Omega_2W_i](D - R_1L_i)^{-1}b, \quad i = 1, ..., \alpha. \end{aligned}$$

If we consider,

$$\ell_M(R_1, R_2, \Omega_1, \Omega_2) = \sum_i E_i\ell_i(R_1, R_2, \Omega_1, \Omega_2) \tag{2.5}$$

and

$$b_M(R_1, R_2, \Omega_1, \Omega_2) = \sum_i E_i b_i(R_1, R_2, \Omega_1, \Omega_2)$$

we can write the method (2.2) as

$$x^{k+1} = \ell_M(R_1, R_2, \Omega_1, \Omega_2)x^k + b_M(R_1, R_2, \Omega_1, \Omega_2) \tag{2.6}$$

If we take $\alpha = 1$, L_1 the strictly lower part of A and U_1 the strictly upper part of A (at this time $W_1 = 0$, is the zero matrix), then the method (2.6) reduces to the unsymmetric modified AOR (USMAOR) method. The Modified AOR (MAOR) method was introduced in [7].

If we consider $D = D_1 - D_2 - D_3$, where D_1 and D_2 are arbitrary diagonal matrices, with det $(D_1) \neq 0$ and det $(D_1 - rD_2) \neq 0, \omega$ and r real parameters $(\omega \neq 0)$ we can obtain the multisplitting method associated to the generalized AOR (MGAOR) method. The Generalized AOR (GAOR) was presented in [8] by Hadjidimos.

For specific values of the parameters, we obtain from (2.6) the multispliting methods associated to the well-known iterative methods, as we can see in the following table, where r_i, ω_i, $(i = 1, 2)$ and δ are real parameters $(\omega_i \neq 0)$, and Δ a positive diagonal matrix.

Remarks

(I) If in methods $(I) - (X)$ we consider $R_j = \Omega_j, j = 1, 2$ or $r_j = \omega_j$, $j = 1, 2$ we obtain the corresponding versions of Modified SOR and SOR methods, respectively.

(II) As it is well known for special values of the parameters, from the methods (XI) and (XII), we can obtain the multisplitting methods associated to the generalized version of the well known iterative methods and the well known iterative methods in the meaning of a single splitting, respectively.

Thus if we consider (ω, r) equal to $(1, 0), (1, 1), (\omega, \omega), (\omega, 0)$ and $(\omega, 1)$ we obtain from (XII) the generalized Jacobi method (GJ) the generalized Gauss-Seidel method (GGS), the generalized simultaneous Jacobi method (GJOR) and the generalized successive overrelaxation method (GSOR) the generalized simultaneous Jacobi method (GJOR) and the generalized extrapolated Gauss Seidel method (GEGS) respectively.

TABLE

Parameters					Iterative methods	
R_1	R_2	Ω_1	Ω_2	α	MUSMAOR (multisplitting method associated to the unsymmetric modified AOR method)	(I)
R_1	R_2	Ω_1	Ω_2	1	USMAOR (unsymmetric modified AOR method)	(II)
R_1	R_1	Ω_1	Ω_1	α	MSMAOR (multisplitting method associated to the symmetric modified AOR method)	(III)
R_1	R_1	Ω_1	Ω_1	1	SMAOR (symmetric modified AOR method)	(IV)
R_1	0	Ω_1	0	α	MMAOR (multisplitting method associated to the modified AOR method)	(V)
R_1	0	Ω_1	0	1	MAOR (modified AOR method)	(VI)
$r_1 I$	$r_2 I$	$\omega_1 I$	$\omega_2 I$	α	MUSAOR (multisplitting method associated to the unsymmetric AOR method)	(VII)
$r_1 I$	$r_2 I$	$\omega_1 I$	$\omega_2 I$	I	USAOR (unsymmetric AOR method)	$(VIII)$
$r_1 I$	$r_1 I$	$\omega_1 I$	$\omega_1 I$	α	MUSAOR (multisplitting method associated to the symmetric AOR method)	(IX)
$r_1 I$	$r_1 I$	$\omega_1 I$	$\omega_1 I$	1	SAOR (symmetric AOR method)	(X)
$r(D_1 - rD_2)^{-1}D$	0	$\omega(D_1 - rD_2)^{-1}D$	0	α	MGAOR (multisplitting method associated to the generalized AOR method)	(XI)
$r(D_1 - rD_2)^{-1}D$	0	$\omega(D_1 - rD_2)^{-1}D$	0	1	GAOR (generalized AOR method)	(XII)
$\delta\Delta$	0	Δ	0	α	MJames (multisplitting method associated to the James' method)	$(XIII)$
$\delta\Delta$	0	Δ	0	1	(James' method)	(XIV)

3. Notation and auxiliary results

In this section we give some basic notation and known results, needed in the sequel.

Let $A, B \in R^{n,n}$. We write $A \leq B$ $(<)$ if $a_{ij} \leq b_{ij}$ $(<)$ holds for all entries of $A = (a_{ij})$ and $B = (b_{ij})$, $i, j = 1, ..., n$; calling $A \in R^{n,n}$ non-negative if $a_{ij} \leq 0$, $i, j = 1, ..., n$.

In particular, the vector $x \in R^n$ is positive (writing $x > 0$) if all its entries are positive.

The absolute value of $A \in R^{n,n}$ defined by $|A| = (|a_{ij}|)$, $(i, j = 1, 2, ..., n)$ is a non-negative matrix satisfying $|AB| \leq |A||B|$, for $B \in R^{n,n}$.

With this notation we introduce some definitions and lemmas:

Definition 1. *[9] The comparison matrix (Ostrowski matrix) of $A \in R^{n,n}$ is denoted by $\langle A \rangle = (\langle a_{ij} \rangle), i, j = 1, 2, ..., n$, where*

$$\langle a_{ij} \rangle = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}$$

Definition 2. *[10] A real $n \times n$ matrix $A = (a_{ij})$, $i, j = 1, 2, ..., n$, is an $\mathcal{M}-$ matrix if $a_{ij} \leq 0$ for $i \neq j$, A is nonsingular and $A^{-1} \geq 0$*

Definition 3. *[10] A matrix $A \in R^{n,n}$ is an $H-$matrix if $\langle A \rangle$ is an $\mathcal{M}-$ matrix.*

Lemma 1. *[10] Let $A \in R^{n,n}$ be a non-negative irreducible matrix. Then, the spectral radius $\rho(A)$ of A is an eigenvalue of A and the eigenvector x associated to $\rho(A)$ satisfies $x > 0$.*

Lemma 2. *[11] Let A, B be $n \times n$ $\mathcal{M}-$ matrices, $D = diag(A)$ and $C \in R^{n,n}$. Then,*

(i) D is nonsingular and D, D^{-1} are non-negative matrices with positive diagonal entries.

(ii) $A \leq B \Rightarrow B^{-1} \leq A^{-1}$

(iii) $A \leq C \leq D \Rightarrow C$ is an $\mathcal{M}-$matrix.

Lemma 3. *[10] Let $A, B \in R^{n,n}$ verifying $|A| \leq B$. Then $\rho(A) \leq \rho(B)$.*

Lemma 4. *[2] Let $A \in R^{n,n}$ be an $H-$matrix, $D = diag(A)$, and $A = D - B$. Then*

(i) A is nonsingular

(ii) $|A^{-1}| \leq \langle A\rangle^{-1}$

(iii) $|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.

4. Convergence results

Let us consider,

$$\ell_M(R_1, R_2, \Omega_1, \Omega_2) = \sum_i E_i \ell_i(R_1, R_2, \Omega_1, \Omega_2) \tag{4.1}$$

$$= \sum_i E_i[M_i(R_2)]^{-1} N_i(R_2, \Omega_2)[M_i(R_1)]^{-1} N_i(R_1, \Omega_1)$$

where

$$M_i(R_1) = D - R_1 L_i$$
$$M_i(R_2) = D - R_2 U_i$$
$$N_i(R_1, \Omega_1) = (I - \Omega_1)D + (\Omega_1 - R_1)L_i + \Omega_1(U_i + W_i)$$
$$N_i(R_2, \Omega_2) = (I - \Omega_2)D + (\Omega_2 - R_2)U_i + \Omega_2(L_i + W_i),\ i = 1, ..., \alpha$$

In order to prove the convergence of MUSMAOR method, we have the following theorem:

Theorem 1. *Let $A \in R^{n,n}$ be an $H-$matrix, $\ell_M(R_1, R_2, \Omega_1, \Omega_2)$ the iteration matrix of the multisplitting method associated to the USMAOR method, given by (2.5). Then,*

$$\rho(\ell_M(R_1, R_2, \Omega_1, \Omega_2)) < 1$$

if

$$R_j \geq 0, \Omega_j \geq R_j,\ 0 < \omega_j^l < \frac{2}{1 + r(|D|^{-1}|B|)},\ \ l = 1, 2, ..., n;\ j = 1, 2, \tag{4.2}$$

where ω_j^l are the diagonal entries of the matrix Ω_j and $B = L_i + U_i + W_i$, $i = 1, ..., \alpha$.

Proof. As the matrix A is an $H-$matrix, the matrices $(D - R_1L_i)$ and $(D - R_2U_i)$ are also $H-$matrices for $i = 1, 2, ..., \alpha$, since L_i and U_i are strictly the lower and upper triangular matrices respectively. Thus, from Lemma 4, we have

$$|M_i(R_1)^{-1}| = |(D - R_1L_i)^{-1}| \leq \langle D - R_1L_i\rangle^{-1} = (|D| - R_1|L_i|)^{-1}$$

and,

$$|M_i(R_2)^{-1}| = |(D - R_2L_i)^{-1}| \leq \langle D - R_2L_i\rangle^{-1} = (|D| - R_2|U_i|)^{-1}$$

If we denote the matrices $|D| - R_1|L_i|$ and $|D| - R_2|U_i|$ by $M_i(R_1)$ and $M_i(R_2)$, respectively, we have:

$$M_i(R_1) \leq |D| \text{ and } M_i(R_2) \leq |D|,\ i = 1, 2, ..., \alpha. \tag{4.3}$$

Therefore, by using Lemma 2, we have

$$M_i(R_1)^{-1} \geq |D|^{-1} \text{ and } M_i(R_2)^{-1} \geq |D|^{-1},\ i = 1, 2, ..., \alpha. \tag{4.4}$$

Following the proof of Theorem 1 [4] we have,

$$|N_i(R_1, \Omega_1)| \leq N_i(R_1, \Omega_1) \text{ and } N_i(R_2, \Omega_2)| \leq N_i(R_2, \Omega_2) \tag{4.5}$$

where,

$$N_i(R_1, \Omega_1) = (|I - \Omega_1| - I)|D| + \Omega_1(|L_1| + |U_i + |W_i)| + (|D| - R_1|L_i|) \tag{4.6}$$

and,

$$N_i(R_2, \Omega_2) = (|I - \Omega_2| - I)|D| + \Omega_2(|L_1| + |U_i| + |W_i|) + (|D| - R_2|L_i|) \tag{4.7}$$

Therefore we can write:

$$N_i(R_j, \Omega_j) = M_i(R_j) - (I - |I - \Omega_j|)|D| - \Omega_j|B|,\ \ i = 1, 2, ..., \alpha,\ j = 1, 2, \tag{4.8}$$

Thus from (4.1) we have:

$$|\ell_M(R_1, R_2, \Omega_1, \Omega_2)| \leq \sum E_i[M_j(R_2)]^{-1}N_i(R_2, \Omega_2)[M_i(R_1)]^{-1}N_i(R_1, \Omega_1). \tag{4.9}$$

Therefore from (4.3), (4.4), and (4.8) we can write:

$$|\ell_M(R_1, R_2, \Omega_1, \Omega_2)| \leq H(\Omega_2)H(\Omega_1) \tag{4.10}$$

with,

$$H(\Omega_j) = |I - \Omega_j| + \Omega_j|D|^{-1}B, \; j = 1, 2, \tag{4.11}$$

From the value ranges of Ω_j ($j = 1, 2$) given by (4.2)

$$\rho(H(\Omega_j)) < 1. \tag{4.12}$$

As the matrices $H(\Omega_j)$, $j = 1, 2$ are non-negative, the matrices

$$H_\epsilon(\Omega_j) = |I - \Omega_j| + \Omega_j(|D|^{-1}|B| + \epsilon ee^T), \; j = 1, 2, \tag{4.13}$$

where $e^T = [1, 1, ..., 1]^T$, are non-negative and irreducible for any $\epsilon > 0$. As $\rho(H(\Omega_j)) < 1$, $j = 1, 2$, by the continuity of the spectral radius we obtain $\rho(H_\epsilon(\Omega_j)) < 1$, $j = 1, 2$ for sufficiently small ϵ.

Now, if we consider the matrices:

$$H_\epsilon^1(\Omega_j) = \begin{cases} I - \min_l w_j^l(I - J_\epsilon) & \text{if } 0 < w_j^l < 1, \\ \max_l w_j^l(I + J_\epsilon) - I & \text{if } 1 \leq w_j^l < \dfrac{2}{1 + \rho(|D|^{-1}|B|)}, \end{cases} \tag{4.14}$$

$l = 1, ..., n$, $j = 1, 2$, where $J_\epsilon = |D|^{-1}|B| + \epsilon ee^T$, we can set that

$$H_\epsilon(\Omega_j) \leq H_\epsilon^1(\Omega_j) \;\text{ and }\; \rho(H_\epsilon^1(\Omega_j)) < 1, \; j = 1, 2 \tag{4.15}$$

and by (4.10) and (4.15) we can write,

$$|\ell_M(R_1, R_2, \Omega_1, \Omega_2)| \leq H_\epsilon^1(\Omega_2)H_\epsilon^1(\Omega_1).$$

As the matrices $H_\epsilon^1(\Omega_1)$ and $H_\epsilon^1(\Omega_2)$ have a common eigenvector set, and by using Lemma 1, we have:

$$|\ell_M(R_1, R_2, \Omega_1, \Omega_2)|x_\epsilon \leq \rho(H_\epsilon^1(\Omega_2))\rho(H_\epsilon^1(\Omega_1))x_\epsilon$$

Thus, using (4.15), we obtain

$$|\ell_M(R_1, R_2, \Omega_1, \Omega_2)|x_\epsilon \leq x_\epsilon$$

Therefore by [10] we have $\rho(\ell_M(R_1, R_2, \Omega_1, \Omega_2) < 1$. □

In the next theorem we establish sufficient convergence conditions for extrapolated method of the MUSMAOR method given by

$$x^{k+1} = \ell_{RM}(R_1, R_2, \Omega_1, \Omega_2, \beta)x^k + b_{RM}(R_1, R_2, \Omega_1, \Omega_2, \beta), k = 0, 1, 2, ... \tag{4.16}$$

where

$$b_{RM}(R_1, R_2, \Omega_1, \Omega_2, \beta) = \beta b_M(R_1, R_2, \Omega_1, \Omega_2)$$

and

$$\ell_{RM}(R_1, R_2, \Omega_1, \Omega_2, \beta) = \beta \ell_M(R_1, R_2, \Omega_1, \Omega_2) + (1 - \beta)I \tag{4.17}$$

with β a real parameter and ℓ_M and b_M defined by (2.5).

Theorem 2. *Let $A \in R^{n,n}$ an $H-$matrix. Then,*

$$\rho(\ell_{RM}(R_1, R_2, \Omega_1, \Omega_2, \beta)) < 1 \quad \textit{if} \;\; 0 \le R_j, \; R_j \le \Omega_j,$$

and

$$0 < \Omega_j < \frac{2}{1 + \rho(|D|^{-1}|B|)} I, \; j = 1, 2, \; 0 < \beta < \frac{2}{1 + \rho(\ell_M(R_1, R_2, \Omega_1, \Omega_2))}$$

Proof. It follows from the extrapolation theorem [12]. □

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