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SPECTRAL ANALYSIS IN CONNECTION WITH ITERATIVE SOLUTION OF CONVECTION-DIFFUSION EQUATION

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Abstract

In order to solve a convection-diffusion equation we use the combination of the spline difference scheme and the central difference scheme. The obtained linear system can not be solved by using classical iterative methods, because they are not convergent. So, we use the spectrum enveloping technique to define convergent iteration procedure.

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1. Introduction

The convection-diffusion equation

(1) $$\Delta u + R\frac{\partial u}{\partial x} = f(x,u),$$

where Δ represents the Laplacian operator and R represents the Reynolds number, is considered. We discretise this differential equation using the cubic spline difference scheme, which is derived in [4], along x - axis and central difference scheme along y - axis. The spline difference scheme derived in [4] for the problem:

$$u'' + Ru' = f(x);\ \ u(0) = \alpha_0,\ \ u(1) = \alpha_1, \tag{2}$$

has the following form:

$$r^- u_{j-1} + r^c u_j + r^+ u_{j+1} = s^- f_{j-1} + s^c f_j + s^+ f_{j+1},\ \ j = 1, 2, \ldots, n, \tag{3}$$

$$u_0 = \alpha_0,\ \ u_{n+1} = \alpha_1, \tag{4}$$

where

$$r^- = -(2 - hR),\ r^+ = -(2 + hR),\ r^c = 4,$$
$$s^- = \frac{-h^2}{3},\ s^+ = \frac{-h^2}{3},\ \ s^c = -\frac{4h^2}{3},\ h = \frac{1}{n+1}.$$

Let u_j^m be the approximation for $u(jh, mh)$. Then using two mentioned schemes we obtain the following scheme for equation (1) :

$$A_j^m + B_j^m + C_j^m = F_j^m \tag{5}$$

where

$$A_j^m = (s^-/h^2)u_{j-1}^{m-1} + (r^- - 2s^-/h^2)u_{j-1}^m + (s^-/h^2)u_{j-1}^{m+1},$$
$$B_j^m = (s^c/h^2)u_j^{m-1} + (r^c - 2s^c/h^2)u_j^m + (s^c/h^2)u_j^{m+1},$$
$$C_j^m = (s^+/h^2)u_{j+1}^{m-1} + (r^+ - 2s^+/h^2)u_{j+1}^m + (s^+/h^2)u_{j+1}^{m+1},$$
$$F_j^m = s^- f_{j-1}^m + s^c f_j^m + s^+ f_{j+1}^m.$$

Equation (5) is defined on a set of n^2 mesh points inside some two dimensional domain Ω. Let w represents the vector of discrete values $u(x, y)$ at the n^2 mesh points. The vector w can be obtained from the linear system

$$Aw = b, \tag{6}$$

where A is n^2 x n^2 coefficient matrix of the following form:

$$A = \begin{bmatrix} P & Q & & & & & \\ Q & P & Q & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & Q & P & Q \\ & & & & & Q & P \end{bmatrix},$$

$$P = tridiag(p^-, \underline{p^c}, p^+)\,,$$

$$Q = tridiag(q^-, \underline{q^c}, q^+)\,,$$

$$p^c = 20/3, \;\; p^- = -4/3 + hR, \;\; p^+ = -4/3 - hR,$$

$$q^c = s^c/h^2, \;\; q^- = s^-/h^2, \;\; q^+ = s^+/h^2, \;\; h = 1/(n+1)$$

and vector b is an n^2-dimensional vector, which depends on the boundary conditions given on the boundary of the domain Ω.

The symbol $tridiag(\alpha, \underline{\beta}, \gamma)$ denotes tridiagonal matrix with diagonal entries β, entries above diagonal γ and entries under diagonal α.

2. Convergence analysis

The Jacobi method for solving sistem (6) has the iteration matrix B_J of order n^2 of the following form:

$$B_J = \begin{bmatrix} \tilde{P} & \tilde{Q} & & & & \\ \tilde{Q} & \tilde{P} & \tilde{Q} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \tilde{Q} & \tilde{P} & \tilde{Q} \\ & & & & \tilde{P} & \tilde{Q} \end{bmatrix},$$

$$\tilde{Q} = -\frac{1}{p^c}Q, \quad \tilde{P} = E - \frac{1}{p^c}P,$$

where E is identity matrix of order n. It is well known that the Jacobi method is convergent for each start vector if and only if $\rho(B_J) < 1$.

As we shall see, $\rho(B_J)$ depends on the mesh Reynolds number $Rh/2$, and it increases while R is increasing. So the Jacobi method will not converge when the Reynolds number is large. In order to obtain a new iteration method that is convergent for all values of the mesh Reynolds number, we shall find an ellipse in the complex plane with major axis on the imaginary axis and minor axis on the real interval (-1,1), which envelops the eigenvalue spectrum of B_J. After that we shall apply the two-step iterative method proposed by de Pillis, [3], and used by Gupta,[1].

Let us estimate $\rho(B_J)$. In fact, the matrices $\tilde{P}$ and $\tilde{Q}$ are

$$\tilde{P} = \frac{1}{20} tridiag(4 - 3hR, \underline{0}, 4 + 3hR), \;\; \tilde{Q} = \frac{1}{20} tridiag(1, \underline{4}, 1).$$

Let us suppose $n > 1$ and denote $N = n^2$. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of the matrix B_J, $\mu_1 \geq \ldots \geq \mu_N$ the eigenvalues of the matrix $C = (B_J + B_J^T)/2$ and $\nu_1 \geq \ldots \geq \nu_N$ the eigenvalues of the matrix $D = (B_J - B_J^T)/2i$ $(i^2 = -1)$. It is wel known ([2]) that for all $k = 1, 2, \ldots, n$

$$\mu_n \leq Re(\lambda_k) \leq \mu_1, \quad \nu_n \leq Im(\lambda_k) \leq \nu_1.$$

In our case the matrices C and D are:

$$C = tridiag(C_0, \underline{C_d}, C_0),$$

where this notation one should use in the block sence,

$$C_d = \frac{1}{5} tridiag(1, \underline{0}, 1), \;\; C_0 = \frac{1}{20} tridiag(1, \underline{4}, 1),$$

$$D = \frac{3hR}{20i} tridiag(0, \underline{G}, 0), \;\; G = tridiag(-1, \underline{0}, 1).$$

Now, we have $||C||_\infty = 1$ and, because of that, $-1 \leq Re(\lambda_k) \leq 1$ for all eigenvalues λ_k of the Jacobi matrix.

Obviously, $||G||_\infty \leq 2$, so that

$$\rho(D) \leq \frac{3hR}{10}, \;\; |Im(\lambda_k)| \leq \frac{3hR}{10}, \;\; k = 1, 2, \ldots, N.$$

It is easy to see that the imaginary parts of the eigenvalues λ_k increase while the Reynolds number R increases. So, the Jacobi method will not converge when the Reynolds number is large.

There is a possibility to define a new iterative method which will be convergent one. Such method is two-step iterative method proposed by Pillis [3] and used by Gupta [1].

In order to formulate the mentioned two-step iterative method, we should find an ellipse in the complex plane with mayor axis on the imaginary axis and minor axis on the real interval $(-1, 1)$, which envelops the eigenvalue spectrum of B_J.

First of all, we shall find the number $m < 1$, such that

$$|Re(\lambda_k)| \leq \rho(C) \leq m$$

(λ_k are the eigenvalues of the Jacobi matrix).

Let us denote by $diag(A_1, A_2, ..., A_n)$ block diagonal matrix with blocks $A_1, A_2, ..., A_n$ on it's diagonal. Now, we define

$$W = diag(\Gamma, \Gamma, ..., \Gamma),$$

$\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, where $\gamma_k = 2 - (k/n)^2, \;\; k = 1, 2, ..., n$.

The matrix W is an $n^2 \times n^2$ regular matrix. Matrices C and $W^{-1}CW$ are similar , so we have

$$\rho(C) = \rho(W^{-1}CW) \leq ||W^{-1}CW||_\infty = 1 - 3/(10n^2 - 20).$$

Finally, we are able to construct the enclosing ellipse, as follows.

We choose $m = 1 - 2/(10n^2 - 20)$ as the minor axis and then we choose major axis M so that $(X/m)^2 + (Y/M)^2 < 1$, for

$$X = 1 - 3/(10n^2 - 20), \;\; Y = 3hR/10.$$

For example, we can allways choose $M = 3R/2$.

The enveloping ellipse is given by

$$(x/m)^2 + (y/M)^2 = 1.$$

Now (see [1]) we can define two constants γ and δ, such that

$$\gamma = (m - M)/(m + M)$$

and δ is the unique root in $(0, 1)$ of the quadratic equation

$$(M + m)/(1 + \gamma\delta^2) = 2\delta.$$

The convergent iterative method is defined by

$$x^{k+2} = (1 + \gamma\delta^2)B_J x^{k+1} - \gamma\delta^2 x^k + \frac{3}{20}(1 + \gamma\delta^2)b, \;\; k \geq 0. \tag{7}$$

De Pillis showed that the sequence defined by (7) converges whenever all real parts of the eigenvalues of the Jacobi matrix B_J are in the interval $(-1, 1)$. In our case this condition is satisfied, so we can conclude that the iterative method defined by (7) is convergent one.

In the following table we shall show the values for m and M for some special choices of n and R.

R	10	100	1000
M	15	150	1500

n	20	50	100
m	0.999497	0.9992	0.9998

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