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# UNDECIDABLE VARIETIES WITH SOLVABLE WORD PROBLEMS – II

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**Abstract**

The purpose of this paper is to present a new example of a recursively based semigroup variety (of simpler type than the examples, described in earlier papers concerning this field), having solvable local word problem, but unsolvable equational theory.

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## 1. Introduction

Given an algebraic language $\mathcal{L}$ and a set $\Sigma$ of identities, different decision problems concerning $\Sigma$ may arise. Generally, one can ask if the sets of all first–order, implicational or equational consequences of $\Sigma$ are recursive. If so, we say that the elementary, implicational, equational theory based on $\Sigma$ are decidable.

For example, Abelian groups and Boolean algebras appear to have decidable elementary theory. Obviously, decidability of elementary theory yields decidability of implicational theory, and that decidability of equational theory. Decidable equational theories include commutative semigroups, groups, lattices, etc. On the other hand, modular lattices and relation algebras have undecidable equational theories.

An another kind of decision problems in algebra are *word problems.* A *presentation* is a pair $(G, R)$, where $G$ is a set of new constant symbols, extending $\mathcal{L}$ to $\mathcal{L}_G = \mathcal{L} \cup G$, and $R$ is a set of equations over $\mathcal{L}_G$ in which no variables appear. The presentation is finite, if $G$ and $R$ are both finite. The word problem for $(G, R)$ over $\Sigma$ is *solvable* iff the set of equational consequences of $\Sigma \cup R$ without variables is recursive, i.e. iff there is an algorithm to decide whether any two words in the language $\mathcal{L}_G$ having no variables are equal.

An algebra **A** is *presented* by $(G, R)$, iff **A** is isomorphic to the $\mathcal{L}$-reduct of the 0-rank free algebra of the variety, generated by $\Sigma \cup R$, or equivalently iff it is isomorphic to $\mathbf{F}_{\mathcal{V}}(G)/\theta_R$, where $\mathcal{V}$ is the variety generated by $\Sigma$, and

$$\theta_R = \{(p, q) | \Sigma \cup R \vdash p \approx q\},$$

is a congruence on $\mathbf{F}_{\mathcal{V}}(G)$. Denote such **A** by $\mathbf{A} = \mathcal{P}_{\mathcal{V}}(G, R)$. Now, the word problem for **A** is the word problem for $(G, R)$.

By investigating word problems for varieties of algebras, one is concerned with two questions:

(1) is the word problem solvable for each finitely presented algebra $\mathbf{A} = \mathcal{P}_{\mathcal{V}}(G, R)$?

(2) is there a universal algorithm which, given a finite presentation $(G, R)$, solves the word problem for $\mathbf{A} = \mathcal{P}_{\mathcal{V}}(G, R)$?

If the answer to (1) is positive, we say that $\mathcal{V}$ has *solvable local word problem* (the word *local* is usually omited). If (2) has a positive answer, we say that $\mathcal{V}$ has *solvable global* (or *uniformly solvable*) *word problem.*

One can prove that decidability of the implicational theory based on $\Sigma$ and the global word problem for $\mathcal{V} = mod(\Sigma)$ are equivalent.

In this paper, we are going to present semigroup varieties of the types (2,1,0) and (2,1) with solvable word problems having undecidable equational theories (which implies the unsolvability of the global word problems).

Examples of varieties with this property were presented earlier in the papers of Wells [11],[12],[13], Mekler, Nelson, Shelah [9], Crvenković, Delić

[3],[4] and Crvenković, Dolinka [5],[6]. This paper is a cotribution to the topic. Of course, reader will immediately note that we already constructed examples of varieties of these types in [5]. However, one may wish to construct undecidable semigroup varieties having solvable word problem, but undecidable equational theory. Recall that all earlier examples, except [5] and [6], listed above, were semigroups with operators and/or constants. Therefore, for some "traditional" reasons, we give this one "extra" example.

## 2. Example of a semigroup variety of the type (2,1,0)

In the sequel, $\varphi$ will be a primitive recursive function, $X = \{\varphi(k) | k \in \mathbf{N}\}$ nonrecursive recursively enumerable set with $1 \notin X$, where $\mathbf{N} = \{1, 2, \ldots\}$.

Consider the algebraic language $\{\cdot, f, 0\}$ of the type (2,1,0) and the following identities in this language:

(1) $$(xy)z \approx x(yz),$$
(2) $$x^2 \approx 0,$$
(3) $$x \cdot 0 \approx 0,$$
(4) $$0 \cdot x \approx 0,$$
(5) $$xyzu \approx xzyu,$$
(6) $$f(0) \approx 0,$$
(7) $$f(f(x)) \approx 0,$$
(8) $$f(x)y \approx 0,$$
(9) $$xf(xy) \approx 0,$$
(10) $$xyf(z) \approx 0,$$
(11) $$xf(yf(zu)) \approx xf(zf(yu)),$$
(12) $$f(xyz) \approx 0,$$
(13) $$xf(x) \approx 0,$$
(14) $$x_1 f(x_2 f(\ldots f(x_{\varphi(n)} f(x_1)) \ldots)) \approx f^n(0),\ n \in \mathbf{N}.$$

Let $\mathcal{V}_1$ dentotes the variety generated by the identities (1)–(13). Variety $\mathcal{V}$ will be its subvariety, which, except (1)–(13) satisfies also the identity (14). Of course, the set of identities listed above is recursive.

Note that this set of identities is, in fact, a kind of "imitation in semigroups" of the groupoid variety from [6]. The unary symbol simulates non-associativity and the brackets of groupoid terms. We are going to prove:

**Theorem 2.1.** *Variety $\mathcal{V}$ has solvable word problem and undecidable equational theory.*

## 3. Solving the word problem for $\mathcal{V}$

We are going to define an algebra $\mathbf{S} = (S, \cdot, \phi, 0)$ of the type (2,1,0), where $S$ consists of some finite sequences over $\mathbf{N_0} = \mathbf{N} \cup \{0\}$ and $\emptyset$ – empty sequence. We say that the sequence $a_1, \dots, a_k$ is *sorted* if

$$a_1 < \dots < a_k$$

holds. If the sequence $a_1, \dots, a_k$ contains different natural numbers, the sequence, obtained by sorting this sequence, we shall denote by

$$sort(a_1, \dots, a_k).$$

Let $A$ be the set of sequences $(a_1, \dots, a_n)$ satisfying the conditions:

(1) $a_i \neq 0$,

(2) $a_i \neq a_j$ for $1 \leq i < j < n$ or $1 < i < j \leq n$,

(3) the sequence $a_2, \dots, a_{n-1}$ is sorted,

(4) if $n = 2$, then $a_1 \neq a_2$.

Also, we put $\emptyset \in A$.

Let $B$ be the set which contains $A$ as the subset, and also the following sequences:

$$(b_1, 0, \dots, 0, b_k, 0, a_1), \text{ where } (b_1, \dots, b_k, a_1) \in A,$$

$$(b_1, 0, \dots, 0, b_k, 0, a_1, a_2), \text{ where } (b_1, \dots, b_k, a_1, a_2) \in A.$$

Finally, define $C$ to be the set of sequences with 0 as the first element, while the rest of the sequence belongs to $A$, and having length $\leq 2$ or it belongs to $B \setminus A$. Now, let $S = B \cup C$. Define the unary operation $\phi$:

$$\phi(a) = \begin{cases} (0, a) & \text{if } a \in B, a \neq \emptyset, |a| \leq 2 \text{ if } a \in A \\ \emptyset & \text{if } a = \emptyset \text{ or } a \in C \text{ or } a \in A, |a| \geq 3 \end{cases}$$

We have to define the binary operation. We are going to do it in five steps:

**1.** If $a = \emptyset$ or $b = \emptyset$, the result is $\emptyset$.

**2.** If $a \in (B \setminus A) \cup C$, then $ab = \emptyset$.

**3.** If $b \in B \setminus A$, then $ab = \emptyset$.

**4.** If $b \in C$ and $a$ has at least two elements, then $ab = \emptyset$. If $a$ is a singleton and

$$b = (0, b_1, 0, \ldots, 0, b_k, 0, c_1, \ldots, c_n),$$

where $n \leq 2$, then define $ab = (a_1, 0, b'_1, \ldots)$, if the conditions $a_1 \neq b_i, 1 \leq i \leq n$, $b_1 \neq c_n$ and $a_1 \neq c_1$ for $n = 2$, are fullfiled. By $(b'_1, \ldots)$ we denoted $sort(b_1, \ldots, b_k)$ for $n = 1$, or $sort(b_1, \ldots, b_k, c_1)$ for $n = 2$. If some of theese conditions fails, define $ab = \emptyset$.

**5.** If $a, b \in A$, $a, b \neq \emptyset$, the result of $ab$ is

$$(a_1, sort(a_2, \ldots, a_n, b_1, \ldots, b_{k-1}), b_k),$$

under the following conditions: $b_k \notin \{a_2, \ldots, a_n\}$, $a_1 \notin \{b_1, \ldots, b_{k-1}\}$, $a_1 \neq a_n$ if $n \neq 1$, $b_1 \neq b_k$ if $k \neq 1$ and $a_1 \neq b_k$ if $n = k = 1$. Otherwise, the result is $\emptyset$.

**Lemma 3.1.** $\mathbf{S} \in \mathcal{V}_1$.

*Proof.* We are now checking the axioms:

(1) If at least one of the sequences $x, y, z$ belongs to $B \setminus A$ or if some of them is $\emptyset$ or $x \in C$, the identity is satisfied. If $y \in C$, the r.h.s. equals to $\emptyset$, while $xy \neq \emptyset$ only if $x$ is a singleton, and all other conditions of **4.** hold. But then $xy \in B \setminus A$, so $(xy)z = \emptyset$. The only case remaining is $x, y \in A$, $x, y \neq \emptyset$. If $z \in C$, then automatically $(xy)z = \emptyset$, while $yz = \emptyset$ or $yz \in B \setminus A$, in both cases $x(yz) = \emptyset$. Finally, if $z \in A$, $z \neq \emptyset$, the only possibility when $(xy)z, x(yz)$ are not both $\emptyset$ is when they have the same value

$$(x_1, sort(x_2, \ldots, x_n, y_1, \ldots, y_k, z_1, \ldots, z_{m-1}), z_m),$$

which one easily checkes.

(2) The only nontrivial case is when $x \in A$, which is easy to check.

(3),(4) By the definition of the multiplication.

(5) Nontrivial case is $x, y, z \in A$, $u \in A \cup C$. If $u \in A$, analogously as for (1), the only case when at least one of the sides of the identity differs from $\emptyset$ is when they both have the value

$$(x_1, sort(x_2, \ldots, x_n y_1, \ldots, y_k, z_1, \ldots, z_m, u_1, \ldots, u_{r-1}), u_r).$$

If $u \in C$, neither $xyz$ nor $xzy$ are singletons, so $xyzu = xzyu = \emptyset$.

(6) $\phi(\emptyset) = \emptyset$.

(7) $\phi(x) = \emptyset$ or $\phi(x) \in C$, so $\phi(\phi(x)) = \emptyset$.

(8) For the same reason as in (11), $\phi(x)y = \emptyset$.

(9) Nontrivial case appears only when $x$ is a singleton and $\phi(xy) \in C$. In that case, we have:

$$x\phi(xy) = (x_1)(0, x_1, y_1, \ldots) = \emptyset.$$

(10) $\phi(z) = \emptyset$ or $\phi(z) \in C$, but $xy$ is not a singleton, and therefore $xy\phi(z) = \emptyset$.

(11) We have a nontrivial case when $x, y, z$ are singletons and when $u$ is a singleton or $u \in C$. In this case, it follows:

$$x\phi(y\phi(zu)) = (x_1, 0, v_1, 0, v_2, \ldots) = x\phi(z\phi(yu)),$$

where $u = (u_1)$ or $u = (0, u_1, \ldots)$, and

$$(v_1, v_2, v_3, \ldots) = sort(y_1, z_1, u_1, \ldots).$$

(12) One easily checkes that either $xyz \in A$ and having length $\geq 3$, either $xyz = \emptyset$ or $xyz \in C$.

(13) The identity holds almost trivially if $\phi(x) = \emptyset$ or $x$ is not a singleton. But if $x = (x_1)$, we have $(x_1)(0, x_1) = \emptyset$.□

It is a routine to show that all words of the free algebra $\mathbf{F}_{\mathcal{V}_1}$ over the countable set of generators $\{g_1, g_2, \ldots\}$ are listed bellow:

**1.** $0$, $g_i$, $g_i g_j (i \neq j)$, $g_{k_1} g_{k_2} \cdots g_{k_m} g_k$, where $(k_i)_{i=2}^{m}$ is sorted and $k \neq k_r$ for all $r \geq 2$,

**2.** $f(w)$, $g_i f(w)$, where $w$ is a word of the type **1.** of length $\leq 2$ and $i \neq k_1$,

**3.** $f(g_{k_1} f(\dots f(g_{k_s} f(w))\dots)))$, where is $w$ of the type **1.** with $|w| \leq 2$, and its letters are not among $g_{k_i}$, $i \geq 2$ and if $|w| = 2$, the first letter of $w$ is not $g_{k_1}$.
**4.** $g_i f(w')$, where is $w'$ a word of the type **3.**, $g_i$ is not one of $g_{k_i}$'s, and if $|w| = 2$, $g_i$ differs from the first letter of $w$.

**Lemma 3.2.** *$\mathcal{V}$ has solvable word problem.*

*Proof.* One immediately sees that all listed words are different in $\mathbf{S}$, so $\mathbf{S} \cong \mathbf{F}_{\mathcal{V}_1}$. Now, if we want to obtain the free algebra $\mathbf{F}^n_{\mathcal{V}_1}$ over a set of $n$ free generators, we should consider only the described sequences in which no other number apperars but 0,1,...,$n$. Of course, there is only finitely many of theese sequnces, since no number, with exception of 0, cannot ocurr more than two times in the same sequence, and there is no two consecutive zeros. Therefore, the free algebras $\mathbf{F}^n_{\mathcal{V}_1}$ and $\mathbf{F}^n_{\mathcal{V}}$ are finite, so the word problem of $\mathcal{V}$ is solvable.□

## 4. Undecidability of the equational theory of $\mathcal{V}$

Define a new algebra $\mathbf{S_1} = (S_1, \star, \phi_1, \emptyset)$: the set $S_1$ we obtain from $S$ by excluding sequences $(a_1, 0, \dots, 0, a_m, 0, a_1)$ and $(0, a_1, 0, \dots, 0, a_m, 0, a_1)$, where $m \in X$. The 'star-operation' is defined by:

$$a \star b = \begin{cases} \emptyset & \text{if } a = (a_1), b = (0, b_1, 0, \dots, 0, b_{m-1}, 0, a_1), m \in X \\ ab & \text{otherwise} \end{cases}$$

The unary operation $\phi_1$ is defined as follows: $\phi_1(a) = \emptyset$ if $a \in C \setminus S_1$, otherwise it is $\phi_1(a) = \phi(a)$.

**Lemma 4.1.** *$\mathbf{S_1}$ is a homomorphic image of $\mathbf{S}$.*

*Proof.* Let us define a mapping $\rho : S \mapsto S_1$ by $\rho(a) = a$ if $a \in S_1$, otherwise $\rho(a) = \emptyset$. One immediately sees that $\rho$ is 'onto'. We are going to prove that $\rho$ is a homomorphism, i.e. that the following equalities hold for all $a, b \in S$:

$$\rho(ab) = \rho(a) \star \rho(b),$$

$$\rho(\phi(a)) = \phi_1(\rho(a)).$$

If $\rho(b) = \emptyset$ then $b \in S \setminus S_1$, so we have either $b \in B \setminus A$, $ab = \emptyset$ or $b \in C$, where the second and the last member of the sequence $b$ are equal, so $ab = \emptyset$, and the first equality is true. We have the same conclusion, if $\rho(a) = \emptyset$. Therefore, consider the case $\rho(a) = a$, $\rho(b) = b$, $a, b \in S_1$. If $a$ is not a singleton, then $ab \neq \emptyset$ only in the case, described in **5.**; but then $ab \in A$, $a \star b = ab = \rho(ab)$. Assume $a = (a_1)$. The case $b \in A$ is already resolved, while the case $b \in B \setminus A$ is trivial. It remains $b \in C$. If $a \star b \neq \emptyset$, we have $ab = a \star b \in S_1$, $\rho(ab) = ab$. In the contrary, $ab \notin S_1$, so we have $\rho(ab) = \emptyset = a \star b$.

Let us check the second relation. If $a \in S_1$ it follows $\rho(a) = a$, $\phi_1(\rho(a)) = \phi(a) \in S_1$, and because of that $\rho(\phi(a)) = \phi(a)$, so we are done. In the opposite case, $\rho(a) = \emptyset$, $a \notin S_1$. If $a \in C$, then $\phi(a) = \emptyset$, otherwise $a \in B$, $\phi(a) = (0, a)$, $\rho(\phi(a)) = \emptyset$.□

So, $\rho(\mathbf{S}) = \mathbf{S_1}$, which implies that $\mathbf{S_1}$ satisfies (1)–(13). But, except that, (14) is also true in this algebra. The nontrivial case to check is for the valuation $x_1 = (a_i)$, with $a_i$'s different, when we have ($m \in X$):

$$x_1\phi(x_2\phi(\ldots\phi(x_m\phi(x_1))\ldots))) \notin S_1,$$

and therefore:

$$x_1 \star \phi(x_2 \star \phi(\ldots \star \phi(x_m \star \phi(x_1))\ldots))) = \emptyset.$$

So, we just proved:

**Lemma 4.2.** $\mathbf{S_1} \in \mathcal{V}$.

But if $m \notin X$, the result of the previous expression is

$$(a_1, 0, a_2, 0, \ldots, 0, a_m, 0, a_1) \neq \emptyset.$$

This implies:

$$\mathbf{S_1} \models x_1 f(x_2 f(\ldots f(x_m f(x_1))\ldots))) \approx 0 \text{ iff } m \in X,$$

i.e. we have the following

**Lemma 4.3.**

$$\mathcal{V} \models x_1 f(x_2 f(\ldots f(x_m f(x_1))\ldots))) \approx 0 \textit{ iff } m \in X.$$

So, $Eq(\mathcal{V})$ is undecidable.

## 5. Semigroup variety of the type (2,1)

**Corolary 5.1.** *The variety of semigroups with an operator, defined by the following identities, has solvable word problem and undecidable equational theory:*

$$\begin{array}{rr} (xy)z \approx x(yz), & x^2 \approx y^2, \\ x^2y \approx yx^2 \approx x^2, & xyzu \approx xzyu, \\ f(x^2) \approx x^2, & f(f(x)) \approx x^2, \\ f(x)y \approx x^2, & xf(xy) \approx x^2, \\ xyf(z) \approx x^2, & xf(yf(zu)) \approx xf(zf(yu)), \\ f(xyz) \approx x^2, & xf(x) \approx x^2, \end{array}$$

$$x_1 f(x_2 f(\ldots f(x_{\varphi(n)}) f(x_1)) \ldots)) \approx f(x_n^2), \; n \in \mathbf{N}.$$

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