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ON THE HOMOGENEOUS DIFFERENCE EQUATION IN THE FIELD OF MIKUSIŃSKI OPERATORS

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Abstract

A homogeneous difference equation, corresponding to a linear homogeneous second order differential equation with constant coefficients in the field of Mikusiński operators is considered. The solution of that difference equation is constructed and its character analyzed.

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1. Notions and notations

The set of continuous functions $\mathcal{C}$ with supports in $[0,\infty)$, with the usual addition and the multiplication given by the convolution

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \quad t \geq 0,$$

is a ring. By the Titchmarsh theorem, $\mathcal{C}$ has no divisors of zero, hence its quotient field can be defined (see [2]). The elements of this field, the

Mikusiński operator field, $\mathcal{F}$, are called *operators*. They are the quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C},\ 0 \not\equiv g \in \mathcal{C},$$

where the last division is observed in the sense of convolution.

Every continuous function $a = a(t)$, $t \geq 0$, defines a unique operator, $\dfrac{a * g}{g}$, where $g \in \mathcal{C}$ is not identically equal to zero. This operator will be simply denoted by a; then we write $a = \{a(t)\}$ and say that the operator a represents the continuous function $a = a(t)$. Such operators are the *integral operator* l and its powers l^α, $\alpha > 0$, namely

$$l = \{1\}, \quad l^\alpha = \left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}, \quad \alpha > 0.$$

The *identity operator* I is given by the quotient $\dfrac{g}{g}$, $g \in \mathcal{C}$, $g \not\equiv 0$.

Let us denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators representing continuous functions, and by $\mathcal{F}_I$ the subset of $\mathcal{F}$ consisting of the elements of the form γI, for some numerical constant γ.

The inverse operator of the integral operator l is the differential operator s, i.e., $ls = I$. It has the property

$$\{f'(t)\} = sf - f(0)I,$$

where f is continuously differentiable on $[0, +\infty)$, and, more generally,

$$\{f^{(n)}(t)\} = s^n f - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)I,$$

provided that the function f has a continuous $n-$th derivative on $[0, +\infty)$.

Note that neither I, nor the powers of the differential operator s, s^n, $n \in \mathbf{N}$, represent any continuous function.

2. Introduction

In our previous paper [5], we analyzed the character of the solution of the first order operator differential equation with its corresponding difference analogue in the field of Mikusiński operators $\mathcal{F}$ In this paper, we consider the homogeneous second order differential equation

$$Au''(x) + Bu'(x) + Cu(x) = 0, \tag{1}$$

where $A \neq 0$, B and C are operators from the field $\mathcal{F}$, while $u(x)$ is the unknown operator function.

Similarly as was done in [3] and [4], taking instead of $u'(x)$ the quotient

$$\frac{u(x+h) - u(x-h)}{2h} \tag{2}$$

for $h > 0$, and also instead of $u''(x)$ the expression

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \tag{3}$$

we obtain the following homogeneous difference equation in the field $\mathcal{F}$:

$$\begin{aligned} &A\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + B\frac{u(x+h) - u(x-h)}{2h} \\ &\quad + Cu(x) = 0. \end{aligned} \tag{4}$$

If we denote $x_n = x_{n-1} + h$ for $h > 0$, $n = 0, \pm 1, \pm 2, \ldots$, and define the operators u_n by $u_n = u(x_n)$, then equation (4) can be written as

$$au_{n-1} + bu_n + cu_{n+1} = 0. \tag{5}$$

In (5), a, b and c are operators from the field $\mathcal{F}$ given by

$$\begin{aligned} a &= \frac{I}{h^2}\left(A - \frac{Bh}{2}\right), \\ b &= -\frac{I}{h^2}\left(2A - Ch^2\right), \\ c &= \frac{I}{h^2}\left(A + \frac{Bh}{2}\right). \end{aligned} \tag{6}$$

In this paper, we differ two types of coefficients a, b and c given by (6):

I case: $\quad a = a_1 I, \quad b = b_1 I, \quad c = c_1 I;$

II case: $a = a_2 s^{r_1}, \quad b = b_2 s^{r_2}, \quad c = c_2 s^{r_3},$

where a_j, b_j and c_j, $j = 1, 2$, are numerical constants and r_j, $j = 1, 2, 3$, are integers.

Our goal is to construct the solution of difference equation (5) in the field $\mathcal{F}$ for these two cases and analyze its character.

3. Solution of difference equation

The characteristic equation for (5) in the field of Mikusiński operators can be obtained by taking $u_n = \omega^n$, $n \in \mathbf{Z}$, where ω is an operator from $\mathcal{F}$ given by

$$a + b\omega + c\omega^2 = 0. \tag{7}$$

It is tempting to write the solutions of equation (7) as

$$\omega_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}. \tag{8}$$

However, in the theory of Mikusiński operators it does not always hold that the square root of an operator is an operator. From now on, we shall consider only the cases when the last is satisfied, i.e., when the expression $\sqrt{b^2 - 4ac}$, is an operator from $\mathcal{F}$, where a, b and c are the operators given by (6), Then the expressions $\omega_{1,2}$ from (8) will also belong to the field $\mathcal{F}$.

3.1 I case

Lemma 1. *If the operators a, b and c, given by (6), belong to $\mathcal{F}_I$, then the solutions of equation (7) belong to the field $\mathcal{F}_I$, and they can be written as*

$$\omega_1 = q_1 I \quad \textit{and} \quad \omega_2 = q_2 I, \tag{9}$$

where q_1 and q_2 are numerical constants.

Proof. If we denote by: $a = a_1 I$, $b = b_1 I$, $c = c_1 I$, where a_1, b_1 and c_1 are numerical constants, then from (8) it follows that the solutions of equation (7) can be written as

$$\omega_{1,2} = \frac{-b_1 I \pm \sqrt{b_1^2 I - 4a_1 I c_1 I}}{2c_1 I} = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1 c_1}}{2c_1} I = q_{1,2} I. \quad \square$$

Of course, it may happen that the solutions of equation (7) have the form (9) and thus belong to $\mathcal{F}_I$, though the operators a, b and c are not necessarily from $\mathcal{F}_I$. For example, if we take

$$a = a_1 s^r, \quad b = b_1 s^r, \quad c = c_1 s^r,$$

for some $r \in \mathbf{N}$, where a_1, b_1 and c_1 are nonzero numerical constants, then these operators are neither from $\mathcal{F}_c$ nor from $\mathcal{F}_I$. In this case the difference equation (5) can be written as

$$a_1 s^r u_{n-1} + b_1 s^r u_n + c_1 s^r u_{n+1} = 0.$$

Multiplying this equation with l^r we obtain the equation (5) with coefficients from $\mathcal{F}_I$.

The following lemma given for the Mikusiński operators is similar to the case when the coefficients of equation (5) and its solutions u_n are real numbers. We use a method from Chapter 1 in [1].

Lemma 2. *Let the solutions of equation (7) be of the form (9).*

- *Assume the solutions ω_1 and ω_2 are nonequal operators of the form (9) and q_1 and q_2 are nonequal real numbers. Then the solution of equation (5) has the form*

  (10) $$u_n = \alpha q_1^n + \beta q_2^n, \quad n \in \mathbf{Z},$$

  where α and β are arbitrary operators from $\mathcal{F}$.

- *Assume the solutions ω_1 and ω_2 are equal operators and q_1 and q_2 in (9) are equal real numbers. Then the solution of equation (5) has the form*

  (11) $$u_n = \alpha q_1^n + \beta n q_1^n, \quad n \in \mathbf{Z},$$

  where α and β are arbitrary operators from $\mathcal{F}$.

- *Assume q_1 and q_2 in (9) are complex numbers. Then the solution of equation (5) has the form*

  (12) $$u_n = \alpha A_1^n \cos n\phi + \beta A_1^n \sin n\phi, \quad n \in \mathbf{Z},$$

  where ϕ is given by $(\cos\phi)I = -\dfrac{b}{2\sqrt{ac}}$, A_1 is a numerical constant satisfying $A_1 I = \sqrt{\dfrac{a}{c}}$, while α and β are arbitrary operators from $\mathcal{F}$.

Using Lemma 2 and the obvious properties of $\mathcal{F}_I$ and $\mathcal{F}_c$, we can prove now

Theorem 1. *If the solutions of characteristic equation (7) of equation (5) are from $\mathcal{F}_I$ and*

1. *the operators α and β are from $\mathcal{F}_I$, then the solution of equation (5) is from $\mathcal{F}_I$;*
2. *the operators α and β are from $\mathcal{F}_c$, then the solution of equation (5) is from $\mathcal{F}_c$.*

Proof. The product of two operators from $\mathcal{F}_I$ is an operator from $\mathcal{F}_I$, and the product of an operator from $\mathcal{F}_I$ and an operator from $\mathcal{F}_c$ is an operator from $\mathcal{F}_c$. Therefore the statement follows from the formulas (10), (11) and (12). □

3.2 II case

Let us now consider the coefficients of the homogeneous difference equation (5) in the forms

$$a = a_1 s^{r_1}, \quad b = b_1 s^{r_2}, \quad c = c_1 s^{r_3}, \quad r_1, r_2, r_3 \in \mathbf{Z}, \tag{13}$$

and a_1, b_1 and c_1 are nonzero numerical constants. Note that case I is a special case of II ($r_1 = r_2 = r_3 = 0$). Then the characteristic equation for difference equation (5) has the form (7). First we shall give the form of the solution of equation (7).

Lemma 3. *If the coefficients of equation (5) are of the form (13), then the solutions of characteristic equation have the form*

$$\omega_{1,2} = s^{\rho}\left(R_{1,2} I + R_{1,2}^c\right), \quad \rho \in \mathbf{Q}, \tag{14}$$

where $R_{1,2}$ are numerical constants and $R_{1,2}^c$ are operators from $\mathcal{F}_c$.

Proof. From (13) it follows that the characteristic equation (7) can be written as

$$s^{r_1} a_1 + s^{r_2} b_1 \omega + s^{r_3} c_1 \omega^2 = 0.$$

Its solutions have the forms

$$\omega_{1,2} = \frac{-b_1 s^{r_2} \pm \sqrt{b_1^2 s^{2r_2} - 4a_1 c_1 s^{r_1+r_3}}}{2c_1 s^{r_3}}.$$

Let us suppose that in the previous relation it holds $2r_2 = r_1 + r_3$. Then we can write

$$\sqrt{b_1^2 s^{2r_2} - 4a_1 c_1 s^{r_1+r_3}} = s^{r_2}\sqrt{b_1^2 - 4a_1 c_1}.$$

So the solutions of characteristic equation have the form

$$\omega_{1,2} = v_{1,2} s^{r_2-r_3}, \quad r_2 - r_3 \in \mathbf{Z}, \tag{15}$$

where $v_{1,2}$ are numerical constants.

If $2r_2 > r_1 + r_3$, then we have

$$\begin{aligned}\sqrt{b_1^2 s^{2r_2} - 4a_1 c_1 s^{r_1+r_3}} &= s^{r_2} b_1 \left(I - \frac{4a_1 c_1}{b_1} s^{r_1+r_3-2r_2}\right)^{1/2} \\ &= s^{r_2} b_1 \sum_{i=0}^{\infty} \binom{1/2}{j} \left(-\frac{4a_1 c_1}{b_1}\right)^j (l^{2r_2-r_3-r_1})^j.\end{aligned}$$

Since it holds that $2r_2 - r_1 - r_3 > 0$, the solutions of characteristic equation can thus be written in the form

$$\omega_{1,2} = s^{r_2-r_3}(\gamma_{1,2} I + \psi_{1,2}), \quad r_2 - r_3 \in \mathbf{Z}, \tag{16}$$

where $\psi_{1,2}$ are operators from $\mathcal{F}_c$ and $\gamma_{1,2}$ are numerical constants.

If $2r_2 < r_1 + r_3$, then we have

$$\sqrt{b_1^2 s^{2r_2} - 4a_1 c_1 s^{r_1+r_3}} = s^{(r_1+r_3)/2}\, 2\sqrt{a_1 c_1}\left(\frac{b_1^2}{4a_1 c_1} s^{2r_2-r_1-r_3} - I\right)^{1/2}$$

$$= i s^{(r_1+r_3)/2} 2\sqrt{a_1 c_1} \sum_{j=0}^{\infty} \binom{1/2}{j} (-1)^j \left(\frac{b_1^2}{4a_1 c_1}\right)^j l^{(r_1+r_3-2r_2)j}.$$

Since it holds that $r_1 + r_3 - 2r_2 > 0$, the solutions of the characteristic equation can be written in the form

$$\omega_{1,2} = s^{(r_1-r_3)/2}\left(\gamma_{1,2}^1 I + \psi_{1,2}^1\right), \quad \frac{r_1 - r_3}{2} \in \mathbf{Q}, \tag{17}$$

where $\psi_{1,2}^1$ are operators from $\mathcal{F}_c$ and $\gamma_{1,2}^1$ are numerical constants. □

These results give us

Theorem 2.

- *If the solutions of equation (7) are nonequal and belong to $\mathcal{F}_c$, i.e., $\rho < 0$ in (14), then the solution of equation (5) can be written in the form*

  (18) $$u_n = \alpha\omega_1^n + \beta\omega_2^n, \quad n \in \mathbf{Z}.$$

  The solution (18) belongs to $\mathcal{F}_c$ only if $n \in \mathbf{N}$, provided that α and β belong either to $\mathcal{F}_c$ or to $\mathcal{F}_I$.

- *If the solutions of equation (7) are nonequal and both $\frac{I}{\omega_1}$ and $\frac{I}{\omega_2}$ belong to $\mathcal{F}_c$, i.e., $\rho > 0$ in (14), then the solution of equation (7) can be written in the form (18) and it belongs to $\mathcal{F}_c$ only if $-n \in \mathbf{N}$, provided that α and β belong either to $\mathcal{F}_c$ or to $\mathcal{F}_I$.*

- *If the solutions of equation (7) are both equal to ω_1 and belong to $\mathcal{F}_c$, then the solution of equation (5) is of the form*

  $$u_n = \alpha\omega_1^n + \beta n\omega_1^n, \quad n \in \mathbf{Z},$$

  and it belongs to $\mathcal{F}_c$ only if $n \in \mathbf{N}$, provided that α and β belong either to $\mathcal{F}_c$ or to $\mathcal{F}_I$.

Finally, we have

Theorem 3. *Assume the solutions of equation (7) have the forms*

(19) $$\omega_{1,2} = q_{1,2}I + \omega_{1,2}^c,$$

where $q_{1,2}$ are numerical constants and $\omega_{1,2}^c$ are operators from $\mathcal{F}_c$, i.e., $\rho = 0$ in (14). If α and β belong to

1. *$\mathcal{F}_c$, then the solution of (5) has the form (18) and belongs to $\mathcal{F}_c$;*
2. *$\mathcal{F}_I$, then the solution of equation (5) has the form*

   (20) $$u_n = u_{n,1}I + u_{n,c}, \quad n \in \mathbf{Z},$$

   where $u_{n,1}$ are numerical constants and $u_{n,c}$ are operators from $\mathcal{F}_c$.

Proof. Firstly, we suppose that the solutions of equation (7), having the form (19), are nonequal. Then the solution of equation (5), for $n \in \mathbf{N}$, has the form

$$\begin{aligned} u_n &= \alpha(q_1 I + \omega_1^c)^n + \beta(q_2 I + \omega_2^c)^n \\ &= \alpha\left(q_1^n I + \sum_{j=1}^{n} \binom{n}{j} q_1^{n-j} (\omega_1^c)^j\right) + \beta\left(q_2^n I + \sum_{j=1}^{n} \binom{n}{j} q_2^{n-j} (\omega_2^c)^j\right) \\ &= \alpha\left(q_1^n I + (\omega_1^c)^n\right) + \beta\left(q_2^n I + (\omega_2^c)^n\right). \end{aligned}$$

Since $n \in \mathbf{N}$, the operators $(\omega_1^c)^n$ and $(\omega_2^c)^n$ represent continuous functions, we obtain that the solution of equation (5) is of the form (20).

If in relation (19) we have $\omega_1 = \omega_2$, then the solution of equation (5), for $n \in \mathbf{N}$, has the form

$$\begin{aligned} u_n &= \alpha(q_1 I + \omega_1^c)^n + \beta n(q_1 I + \omega_1^c)^n \\ &= \alpha\left(q_1^n I + \sum_{j=1}^{n} \binom{n}{j} q_1^{n-j} (\omega_1^c)^j\right) + \beta n\left(q_1^n I + n\sum_{j=1}^{n} \binom{n}{j} q_1^{n-j} (\omega_1^c)^j\right). \end{aligned}$$

So we obtain formula (20).

In the case when $-n \in \mathbf{N}$, i.e., $n = -n_1$, $n_1 \in \mathbf{N}$, it holds

$$\begin{aligned} u_n &= \alpha \frac{I}{(q_1 I + \omega_1^c)^{n_1}} + \beta \frac{I}{(q_2 I + \omega_2^c)^{n_1}} \\ &= \frac{\alpha}{q_1^{n_1}}\left(I + \sum_{j=1}^{\infty} \left(\frac{(\omega_1^c)^{n_1}}{q_1^{n_1}}\right)^j\right) + \frac{\beta}{q_2^{n_1}}\left(I + \sum_{j=1}^{\infty} \left(\frac{(\omega_2^c)^{n_1}}{q_2^{n_1}}\right)^j\right). \end{aligned}$$

The last two sums are operators representing continuous functions, and thus we have just obtained the relation (20).

The solution in all three cases will belong to $\mathcal{F}_c$ if α and β belong to $\mathcal{F}_c$, while if α and β belong to $\mathcal{F}_I$, we obtain relation (20). □

The proof when $\omega_1 = \omega_2$, for $n \in \mathbf{N}$ is analogous to the previous cases and is thus omitted. □

Similarly as in [5], it can shown that the solution of difference equation (5) can be treated as the approximate solution of differential equation (1).

References

[1] Godunov, S. K., Rabenkii, V. S., Raznostnye shemy (in Russian), Nauka, Moscow 1977.

[2] Mikusiński, J., Operational Calculus, Vol. I, PWN – Polish Scientific Publishers, Warszawa and Pergamon Press, Oxford, 1983.

[3] Takači, Dj., Difference scheme in the field of Mikusiński operators, Univ. u Novom Sadu, Zb. Rad. Prirod.–Mat. Fak. Ser. Mat., Vol. 23, No. 1 (1993), 135–144.

[4] Takači, Dj., Takači, A., An application of difference equation in the field of Mikusiński operators I, Bull. Appl. Math., Technical University of Budapest, 909/93 (1993), 109–118.

[5] Takači, Dj., Takači, A., The character of the solution of difference equations in the field of Mikusiński operators, Publ. Math. Debrecen, Vol. 45 (1994), 379–394.

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