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ON THE CONVOLUTION EQUATIONS OVER THE QUARTER-PLANE

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Abstract

We define generalized function spaces $LG'_0(\mathbf{R}^2_+)$ and $LG'_e(\mathbf{R}^2_+)$ over the quarter-plane and study the convolution type equations by using the Laguerre expansions of their elements in two dimensions.

The applications on partial integro-differential equations of the form

$$f(t,s) = c\varphi(t,s) + \int_0^\infty k_1(t-\tau)\varphi(\tau,s)d\tau + \int_0^\infty k_2(s-\sigma)\varphi(t,\sigma)d\sigma +$$

$$\int_0^\infty \int_0^\infty k(t-\tau, s-\sigma)\varphi(\tau,\sigma)d\tau d\sigma$$

are given.

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1. Introduction

In [6] , [5], [2], and [13], [12] are given the description and basic properties of the space of tempered distributions S'_+ through the expansion of their elements in the Laguerre series, in one dimension. We denoted this space by LG'_0. Similarly, in [6] and [7] we investigated the space LG'_e, which is the type of $expA'$–spaces introduced in [4], whose elements have the Laguerre expansions. Recall, $\{l_n, n \in \mathbf{N}_0\}$ is the orthonormal base of the space $L^2(\mathbf{R}_+)$ ($\mathbf{R}_+ = (0,\infty)$, $\mathbf{N}_0 = \mathbf{N}\bigcup\{0\}$), with the elements $l_n(t) = e^{-t/2}L_n(t)$, where $L_n(t) = \sum_{m=0}^{n}\binom{n}{n-m}\frac{(-t)^m}{m!}$, $t \geq 0,\ n \in \mathbf{N}_0$, be the Laguerre polynomials.

Note that all the definitions and assertions may be given over the first ortant $\mathbf{R}^n_+$. This is important for the applications to partial differential equations. For the sake of simplicity we will consider only the case when $n = 2$.

The paper is organized as follows:

In Section 2 is given the intrinsic description of the basic spaces, via the Laguerre expansions. We give several equivalent definitions of these spaces with respect to the corresponding norms. The definition of the tempered convolution over the quarter plane and approximation formula for it are given in Section 3 and 4, respectively. The convolution equation and the existence of its solution are considered in Section 5. Applications in solving integral equations, existence of the solution and the Laguerre series solution of them are given in Section 6. Generalized error estimate (error estimate in distributional sense) is obtained in Section 7. Finally, we apply this method to solve partial differential equations of special kind in Section 8. Appendix provides the evaluation of some distributions in the Laguerre series for previous use.

2. Basic spaces

Let $\mathbf{R}^2_+ = \mathbf{R}_+ \times \mathbf{R}_+$, $\bar{\mathbf{R}}^2_+ = \bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+$, $\bar{\mathbf{R}}_+ = [0,\infty), \mathbf{N}_0 = \mathbf{N}\bigcup\{0\}$, and let $C^\infty(\mathbf{R}^2_+)$ be the space of smooth functions on $\mathbf{R}^2_+$. Like in one dimension, in two dimensions we have the orthonormal base $\{l_{nm}(x,y),\ n,m \in \mathbf{N}_0\}$, of $L^2(\mathbf{R}^2_+)$,([9], p. 65), where $l_{nm}(x,y) = e^{-(x+y)/2}L_n(x)L_m(y)$, and $L_n(x), L_m(y)$ are the Laguerre polynomials.

The space $LG_0(\mathbf{R}^2_+)$ (resp. $LG_e(\mathbf{R}^2_+)$) is defined as

$$LG_0(\mathbf{R}^2_+) = \text{ proj} \lim_{k\to\infty} L_{(k,k)}(\mathbf{R}^2_+)(\text{resp.} LG_e(\mathbf{R}^2_+) = \text{ proj} \lim_{k\to\infty} L_{e(k,k)}(\mathbf{R}^2_+)),$$

where, for $k_1, k_2 \geq 0$:

$$L_{(k_1,k_2)}(\mathbf{R}^2_+) = \{\phi = \sum_{(n,m)\in\mathbf{N}_0^2} a_{nm} l_n(x) l_m(y) \;\mid\; |||\phi|||_{(k_1,k_2)} < \infty\}$$

$$|||\phi|||_{(k_1,k_2)} = (|a_{00}|^2 + \sum_{m=1}^{\infty} |a_{0m}|^2 m^{2k_1}$$

$$+ \sum_{n=1}^{\infty} |a_{n0}|^2 n^{2k_2} \sum_{(n,m)\in\mathbf{N}} |a_{nm}|^2 n^{2k_1} m^{2k_2})^{1/2} < \infty$$

and

$$L_{e(k_1,k_2)}(\mathbf{R}^2_+) = \{\psi = \sum_{(n,m)\in\mathbf{N}_0^2} b_{nm} l_n(x) l_m(y) \;\mid\; |||\psi|||_{e,(k_1,k_2)} < \infty\},$$

$$|||\phi|||_{e,(k_1,k_2)} = (|a_{00}|^2$$

$$+ \sum_{m=1}^{\infty} |a_{0m}|^2 k_2^{2m} + \sum_{n=1}^{\infty} |a_{n0}|^2 k_1^{2n} + \sum_{(n,m)\in\mathbf{N}} |a_{nm}|^2 k_1^{2n} k_2^{2m})^{1/2} < \infty.$$

The strong duals of these spaces are $LG'_0(\mathbf{R}^2_+)$ and $LG'_e(\mathbf{R}^2_+)$, respectively.

We can define the spaces $LG_0(\mathbf{R}^2_+)$ in the following way. Denote by $\mathcal{R}$ a differential operator of the form $\mathcal{R} = \mathcal{R}_x \mathcal{R}_y$, where

$$\mathcal{R}_t = e^{t/2} D_t t e^{-t} D_t e^{t/2}, t > 0.$$

$\mathcal{R}^0$ is the identity operator and

$$\mathcal{R}^{(k_1,k_2)+(1,0)} = \mathcal{R}_x \mathcal{R}^{(k_1,k_2)}; \; \mathcal{R}^{(k_1,k_2)+(0,1)} = \mathcal{R}_y \mathcal{R}^{(k_1,k_2)}, k_1, k_2 \in \mathbf{N_0}.$$

The space LG_0 is defined as the space of all $\phi \in C^\infty(\mathbf{R}^2_+)$ for which the norms

$$||\phi||_{(k_1,k_2)} = ||\mathcal{R}^{(k_1,k_2)}\phi||_0 = (\int_0^\infty \int_0^\infty |\mathcal{R}_x^{k_1} \mathcal{R}_y^{k_2} \phi(x,y)|^2 dx dy)^{1/2}, k_1, k_2 \in \mathbf{N}_0,$$

are finite and

$$< \mathcal{R}^{(k_1,k_2)}\phi(x,y), l_{nm}(x,y) >=< \phi(x,y), \mathcal{R}^{(k_1,k_2)}l_{nm}(x,y) >$$

$$= (-n)^{k_1}(-m)^{k_2} < \phi(x,y), l_{nm}(x,y) >, \; k_1, k_2 \in \mathbf{N_0}, \;\; n, m \in \mathbf{N_0},$$

where

$$< \phi, \psi >= \int_0^\infty \int_0^\infty \phi(x,y)\bar{\psi}(x,y)dxdy, \;\; \phi, \psi \in L^2(\mathbf{R}_+^2).$$

We have $\mathcal{R}l_{nm}(x,y) = nml_{nm}(x,y), n, m \in \mathbf{N}_0$.

The equivalent and significant definition of the space LG_0 is the following one. This is the space of functions $\phi \in C^\infty(\mathbf{R}_+^2)$ such that for every $p \in \mathbf{N}_0, \rho_p(\varphi) < \infty$, where

$$\rho_p(\phi) = \sup_{\substack{(x,y)\in\mathbf{R}_+^2 \\ |(k_1,k_2)|\le p, |(\alpha,\beta)|\le p}} (x^{k_1}y^{k_2}|\phi^{(\alpha,\beta)}(x,y)|).$$

Moreover, LG_0 is equal to the space

$$\{\phi(x,y) \in C^\infty(\bar{\mathbf{R}}_+^2); \rho_p(\phi) < \infty, \} \;\; p \in \mathbf{N}_0.$$

The proof of the equivalence of the sequence of norms ρ_p and $|||\cdot|||_p, p \in \mathbf{N}_0$ is omitted because it is the same as for the one-dimensional case, (see [2], [5], [12]).

Thus, $f \in LG_0'(\mathbf{R}_+^2)$ iff there are $(k_1, k_2) \in \mathbf{N}_0^2$ and a continuous function F which is bounded by a polynomial and supported by $\bar{\mathbf{R}}_+^2$ such that $f = D^{(k_1,k_2)}F$. (D is the distributional derivative).

3. Definition of convolution

Let $f, g \in LG_0'(\mathbf{R}_+^2)$ be of the form

$$f(x,y) = D_{xy}^{(m_1,m_2)}F(x,y), \; g(x,y) = D_{xy}^{(r_1,r_2)}G(x,y), \; (D_{xy}^{(m_1,m_2)} = D_x^{m_1}D_y^{m_2}),$$

where $m_1, m_2, r_1, r_2 \in \mathbf{N}_0$, F and G are continuous on $\mathbf{R}^2$ bounded by polynomials with $\text{supp}F \subset \bar{\mathbf{R}}_+^2$ $\text{supp}G \subset \bar{\mathbf{R}}_+^2$. Then, we define the convolution as follows:

$$(f * g)(x,y) = D_{xy}^{(m_1,m_2)}D_{xy}^{(r_1,r_2)}(\int_0^x \int_0^y F(t,s)G(x-t, y-s)dtds),$$

where $D_{xy}^{(m_1,m_2)} = D_x^{m_1} D_y^{m_2}$. Since arbitrary elements of $LG_0'(\mathbf{R}_+^2)$ are the finite sums of elements of the quoted forms this implies the definition of convolution in $LG_0'(\mathbf{R}_+^2)$.

Proposition 1.

Let f_n, g_n be sequences in $LG_0'(\mathbf{R}_+^2)$ which converge in $LG_0'(\mathbf{R}_+^2)$ to f and $g \in LG_0'(\mathbf{R}_+^2)$, respectively. Then,

$$f_n * g_n \to f * g, \text{ in } LG_0'(\mathbf{R}_+^2), (n \to \infty).$$

Proof. There exists $k \in \mathbf{N}$ such that

$$f_n(x,y) \to f(x,y), \quad g_n(x,y) \to g(x,y) \text{ in } L'_{(k,k)}.$$

For sufficiently large m_1 and m_2, the function

$$(t,s) \mapsto \frac{1}{m_1!m_2!}\kappa(t,s)(x-t)_+^{m_1-1}(y-s)_+^{m_2-1} \in L_{(k,k)},$$

where $\kappa \in C^\infty, \kappa(t,s) = 1$ for $t,s > -1/2$, $\kappa(t,s) = 0$ for $t < -1, s \in \mathbf{R}$ and $t \in \mathbf{R}, s < -1$.

We have that

$$< f_n(x,y), \kappa(x-t, y-s)\frac{1}{m_1!m_2!}(x-t)_+^{m_1-1}(y-s)_+^{m_2-1} >$$

converges (when $n \to \infty$) to

$$< f(t,s), \kappa(x-t, y-s)\frac{1}{m_1!m_2!}(x-t)_+^{m_1-1}(y-s)_+^{m_2-1} > .$$

Put (for $n \in \mathbf{N}$, $(x,y) \in \mathbf{R}_+^2$),

$$F_n(x,y) = \begin{cases} < f_n(x,y), \kappa(x-t,y-s)\frac{1}{m_1!m_2!}(x-t)_+^{m_1-1}(y-s)_+^{m_2-1} >, \\ 0 \qquad \text{otherwise,} \end{cases}$$

$$F(x,y) = \begin{cases} < f(x,y), \kappa(x-t,y-s)\frac{1}{m_1!m_2!}(x-t)_+^{m_1-1}(y-s)_+^{m_2-1} >, \\ 0 \qquad \text{otherwise.} \end{cases}$$

Since the sequence of norms ρ_p, $p \in \mathbf{N}_0$, is equivalent with the sequence of norms $|||\cdot|||_{(k,k)}$, $k \in \mathbf{N}_0$, the boundedness of the sequence $f_n(x,y)$ in

$L'_{(k,k)}(\mathbf{R}^2_+)$ implies that there are $p \in \mathbf{N}_0$ and new m_1 and m_2 such that for every $n \in \mathbf{N}$ there exists $C > 0$ such that

$$\max\{|F(x,y), F_n(x,y)|\} \leq$$

$$C \sup\{t^{k_1} s^{k_2} |D^{(\alpha,\beta)}_{xy}(\kappa(x-t, y-s)\frac{1}{m_1! m_2!}(x-t)^{m_1-1}_+ (y-s)^{(m_2-1)}_+|,$$

$$-2 \leq t \leq x, -2 \leq s \leq y, |(k_1,k_2)| \leq p,\ |(\alpha,\beta)| \leq p\},$$

i. e. for suitable $C_1 > 0$, $\tilde{m}_1 \in \mathbf{N}$ and $\tilde{m}_2 \in \mathbf{N}$,

$$\max\{|F(x,y), F_n(x,y)|\} \leq C_1 x^{\tilde{m}_1 - 1} y^{\tilde{m}_2 - 1},\ \ (x,y) \in \mathbf{R}^2_+, n \in \mathbf{N}.$$

This implies that $F_n, n \in \mathbf{N}$ and F are continuous functions supported by $\bar{\mathbf{R}}^2_+$ such that

$$D^{(m_1,m_2)}_{xy} F_n(x,y) = f_n(x,y),\ \ D^{(m_1,m_2)}_{xy} F(x,y) = f(x,y),$$

$$F_n(x,y) \to F(x,y), \text{ for every } (x,y) \in \mathbf{R}^2_+$$

and

$$\frac{|F_n(x,y)|}{(1+|x|)^{\tilde{m}_1-1}(1+|y|)^{\tilde{m}_2-1}},$$

$$\frac{|F(x,y)|}{(1+|x|)^{\tilde{m}_1-1}(1+|y|)^{\tilde{m}_2-1}} < C_1,\ \ (x,y) \in \mathbf{R}^2_+,\ n \in \mathbf{N}.$$

Similarly, for $g_n, n \in \mathbf{N}$ and g and some $p_1, p_2, \bar{p}_1, \bar{p}_2 \in \mathbf{N}_0$ we have

$$D^{(\bar{p}_1,\bar{p}_2)}_{xy} G(x,y) = g_n(x,y),\ \ D^{(\bar{p}_1,\bar{p}_2)}_{xy} G(x,y) = g(x,y),$$

$$G_n(x,y) \to G(x,y), \text{ for every } (x,y) \in \mathbf{R}^2_+,\ (n \to \infty)$$

$$\frac{|G_n(x,y)|}{(1+|x|)^{\bar{p}_1-1}(1+|y|)^{\bar{p}_2-1}}, \frac{|G(x,y)|}{(1+|x|)^{\bar{p}_1-1}(1+|y|)^{\bar{p}_2-1}} < \tilde{C}_1,$$

where G_n and G have the same properties as F_n and F. So,

$$(f_n * g_n)(x,y) = D^{(r_1,r_2)}_{xy}(\int_0^x \int_0^y F_n(t,s) G_n(x-t, y-s)dtds),\ \ (x,y) \in \mathbf{R}^2_+,$$

$$(f * g)(x,y) = D^{r_1,r_2}_{xy}(\int_0^x \int_0^y F(t,s) G(x-t, y-s)dtds),$$

where $r_1 = m_1 + p_1$, $r_2 = m_2 + p_2$. By using the Lebesque theorem, it follows

$$\int_0^x \int_0^y F_n(t,s) G_n(x-t, y-s)dtds \to \int_0^x \int_0^y F(t,s) G(x-t, y-s)dtds,$$

in $LG'_0(\mathbf{R}^2_+)$, when $n \to \infty$. This implies the assertion of the theorem. □

4. Approximation of convolution

Proposition 2. *Let* $f = \sum_{nm} a_{nm} l_n l_m$, $g = \sum_{nm} x_{nm} l_n l_m$ *be in* $LG_0'(\mathbf{R}_+^2)$. *Then,* $f * g \in LG_0'(\mathbf{R}_+^2)$ *and*

$$(1)\ (f*g)(x,y) = \sum_{(n,m)\in \mathbf{N}_0^2} \Big(\sum_{p+q=n} \sum_{r+s=m} a_{pr}x_{qs} - \sum_{p+q=n-1} \sum_{r+s=m} a_{pr}x_{qs} -$$

$$\sum_{p+q=n} \sum_{r+s=m-1} a_{pr}x_{qs} + \sum_{p+q=n-1} \sum_{r+s=m-1} a_{pr}x_{qs} \Big) l_n(x) l_m(y),$$

(when $p \bigvee q \bigvee r \bigvee s < 0$ *then* $a_{pr} \cdot x_{qs} = 0$.)

Proof. By Proposition 1, we have $f_p * g_p \to f * g$, $p \to \infty$, where $f_p = \sum_{m,n\leq p}$, $g_p = \sum_{m,n\leq p}$. This implies

$$f * g = \sum_{(n,m)\in \mathbf{N}_0^2}^{\infty} \Big(\sum_{p+q=n} \sum_{r+s=m} a_{pr}x_{qs}(l_p(x)l_r(y) * l_q(x)l_s(y)) \Big).$$

Since

$$L_m(y) * L_n(y) = L_{m+n}(x) - L_{n+m+1}(x), \text{ (see[3])}$$

we have

$$l_p(x)l_r(y) * l_q(x)l_s(y) = (l_{p+q}(x) - l_{p+q+1}(x))(l_{r+s}(y) - l_{r+s+1}(y)) =$$
$$l_n(x)l_m(y) - l_{n+1}(x)l_m(y) - l_n(x)l_{m+1}(y) + l_{n+1}(x)l_{m+1}(y)$$

which implies (1).

For $f, g \in LG_0'(\mathbf{R}_+^2)$ we have

$$(|a_{00}|^2 + \sum_{m=1}^{\infty} |a_{0m}|^2 m^{-2k_2} + \sum_{n=1}^{\infty} |a_{n0}|^2 n^{-2k_1} + \sum_{nm} |a_{nm}|^2 n^{-2k_1} m^{-2k_2}) < \infty$$

and

$$(|x_{00}|^2 + \sum_{m=1}^{\infty} |x_{0m}|^2 m^{-2k_2} + \sum_{n=1}^{\infty} |x_{n0}|^2 n^{-2k_1} + \sum_{nm} |x_{nm}|^2 n^{-2k_1} m^{-2k_2} < \infty.$$

This implies

$$|\sum_{p+q=n} \sum_{r+s=m} a_{pr}x_{qs}| \leq M m^{k_1+1} n^{k_2+1}, n, m \in \mathbf{N}_0,$$

and $f * g \in LG_0'(\mathbf{R}_+^2)$. When $n = 0, m \in \mathbf{N}$ we have

$$| \sum_{r+s=m} a_{0r} x_{0s} | \leq M m^{k_2+1}$$

and when $m = 0, n \in \mathbf{N}$,

$$| \sum_{p+q=n} a_{p0} x_{q0} | \leq M n^{k_1+1}.$$

This implies that Proposition 2 holds. □

Proposition 3. *Let $f, g \in LG_e'(\mathbf{R}_+^2)$ have the Laguerre expansions given in Proposition 2. Then, $f * g$ defined by approximation formula (1) is an element of $LG_e'(\mathbf{R}_+^2)$. Moreover, if $f_n \to f$, $g_n \to g$ in $LG_e'(\mathbf{R}_+^2)$, then $f_n * g_n \to f * g$ in $LG_e'(\mathbf{R}_+^2)$, as $n \to \infty$.*

Proof.

For $f, g \in LG_e'(\mathbf{R}_+^2)$ we have

$$\sum_{nm} |a_{nm}|^2 k_1^{-2n} k_2^{-2m} < \infty, \sum_{nm} |x_{nm}|^2 k_1^{-2n} k_2^{-2m} < \infty$$

where k_1 and k_2 are suitable positive numbers. This implies

$$| \sum_{p+q=n} \sum_{r+s=m} a_{pq} x_{rs} | \leq M k_1^n k_2^m (m+1)(n+1) \leq M k_3^n k_4^m, m, n \in \mathbf{N}_0,$$

where $k_3 > k_1$, $k_4 > k_2$. Let $m = 0, n \in \mathbf{N}$. Then,

$$| \sum_{p+q=n} a_{p0} x_{q0} | \leq M k_1^n (n+1) \leq M k_3^n.$$

When $n = 0, m \in \mathbf{N}$ then

$$| \sum_{r+s=m} a_{0r} x_{0s} | \leq M k_2^m (m+1) \leq m_{k_4}^m.$$

From all that was said above implies that $f * g \in LG_e'(\mathbf{R}_+^2)$.

Let $f^\nu = \sum_{nm} a_{nm}^\nu l_n l_m$, $\nu \in \mathbf{N}$ be a sequence in LG_e' and $f = \sum_{nm} a_{nm} l_n l_m \in LG_e'$.

Then $f^\nu \to f$ if and only if the conditions (i) and (ii) hold: (i) $a^\nu_{nm} \to a_{nm}$, $\nu \to \infty$ for every n, m;

(ii) there exist $k_1 > 0, k_2 > 0$ such that $\sum_{(n,m)\in\mathbf{N}_0} |a^\nu_{nm}|^2 k_1^{-2n} k_2^{-2m} < \infty$.

One can prove that for $f^\nu * g^\nu = \sum_{nm} b^\nu_{nm} l_n l_m$ and $f * g = \sum_{nm} b_{nm} l_n l_m$, (i) and (ii) hold, and this implies that $f^\nu * g^\nu \to f * g$ in $LG'_e(\mathbf{R}^2_+)$, $(\nu \to \infty)$.
□

The first part of the previous proof implies

Corollary 1. *The space* $LG'_e(\mathbf{R}^2_+)$ *is a convolution algebra. Moreover, if* $f = \sum_{nm} b_{nm} l_n l_m$, $g = \sum_{nm} x_{nm} l_n l_m$ *are in* $L'_{e(k_1,k_1)}(\mathbf{R}^2_+)$, $L'_{e(k_2,k_2)}(\mathbf{R}^2_+)$ *respectively, then* $f * g \in LG_{e(\tilde{k_1},\tilde{k_2})}(\mathbf{R}^2_+)$ *for any* $\tilde{k}_1 > k_1, \tilde{k}_2 > k_2$.

5. Convolution equations in $LG'_0(\mathbf{R}^2_+)$.

Consider the convolution equation

$$f * g = h, \tag{2}$$

where $f, h \in LG'_0(\mathbf{R}^2_+)$ and g is unknown. We shall solve this equation by using the Laguerre series expansions of elements $LG'_0(\mathbf{R}^2_+)$ and we find the conditions of the solvability of this equation in $LG'_0(\mathbf{R}^2_+)$.

Let $f = \sum_{nm} a_{nm} l_n l_m$ $g = \sum_{nm} x_{nm} l_n l_m$ and $h = \sum_{nm} c_{nm} l_n l_m$. Applying (1) we obtain the following system of equations:

$$\sum_{p+q=n} \sum_{r+s=m} a_{pr} x_{qs} - \sum_{p+q=n-1} \sum_{r+s=m} a_{pr} x_{qs} - \sum_{p+q=n} \sum_{r+s=m-1} a_{pr} x_{qs} +$$
$$\sum_{p+q=n-1} \sum_{r+s=m-1} a_{pr} x_{qs} = c_{nm}$$

or

$a_{10}x_{00} + a_{00}x_{10} = c_{00} + c_{10}$, $a_{01}x_{00} + a_{00}x_{01} = c_{01} + c_{00}$ $a_{11}x_{00} + a_{10}x_{01} + a_{01}x_{10} + a_{00}x_{11} = c_{00} + c_{10} + c_{01} + c_{11}$. The equivalent form for (2) is

$$\sum_{p+q=n} \sum_{r+s=m} a_{pr} x_{qs} = \sum_{i=0}^{n} \sum_{t=0}^{m} c_{lt}, n > 1, m > 1, \tag{3}$$

whose recursive solution is

$$(4)\quad x_{nm} = \frac{1}{a_{00}}[\sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} - \sum_{p+r=n}\sum_{q+s=m} a_{pr}x_{qs} + a_{00}x_{nm}], n, m \in \mathbf{N}_0.$$

It is obvious that this system is solvable iff $a_{00} \neq 0$.

Proposition 4. *The convolution equation (2) is uniquely solvable in* $LG'_e(\mathbf{R}^2_+)$ *for any* $h \in LG'_e(\mathbf{R}^2_+)$ *iff* $a_{00} \neq 0$.

Proof.

The coefficients of the system satisfy (3). This system is formally solvable since $a_{00} \neq 0$. The power series product

$$\sum_{(i,j)\in\mathbf{N}_0^2} a_{ij}t^iu^j \sum_{(i,j)\in\mathbf{N}_0^2} x_{ij}t^iu^j = \sum_{(i,j)\in\mathbf{N}_0^2} c_{ij}t^iu^j,$$

gives the same relation for coefficients as the convolution $f * g = h$.

Put $a(t,u) = \sum_{(i,j)\in\mathbf{N}_0^2} a_{ij}t^iu^j$, $b(t,u) = \sum_{(i,j)\in\mathbf{N}_0^2} x_{ij}t^iu^j$, and $c(t,u) = \sum_{(i,j)\in\mathbf{N}_0^2} c_{ij}t^iu^j$. Assumption on a_{ij} and c_{ij} imply that $a(t,u)$ and $c(t,u)$ are analytic in some neighbourhood of $(0,0)$. Thus,

$$\frac{1}{a(t,u)}c(t,u) = b(t,u)$$

is analytic in the neighbourhood of zero. This b is the unique solution of $ab = c$. Thus, for some $r_1, r_2 > 0$,

$$|x_{ij}| \leq Cr_1^ir_2^j,\ i,j \in \mathbf{N}_0.$$

This implies that $g \in LG'_0$. □

6. Application on solving integral equations

Existence of the solution in LG'_0.

By using the approximation formula for convolution we shall solve in $LG'_0(\mathbf{R}^2_+)$ the following convolution equation:

$$(5)\qquad F(t,s) * \varphi(t,s) = h(t,s).$$

The following Theorem concerning the existence and the uniqueness of the solution of integro-differential equation in two dimensions is a generalization of the results from [8].

Theorem 1.

1. *Let* $F = c\delta(t,s) + k(t,s)$ *in (5), where* $k \in L^1(\mathbf{R}^2_+)$ *and let* $c \neq \hat{k}(z_1, z_2)$, $(z_1, z_2) \in (\mathbf{R}^2 + i\mathbf{R}^2_+) \bigcup \dot{\mathbf{R}}^2$, *where* $\dot{\mathbf{R}}^2$ *is the completition of* $\mathbf{R}^2$. *Then, there exists the unique solution* $\varphi \in L^1(\mathbf{R})$ *of (5).*

2. *Let* $F = c\delta(t,s) + k_1(t)\delta(s) + k_2(s)\delta(t)$ *in (5) where* $k_1, k_2 \in L^1(\mathbf{R}^2)$ *and* $c \neq [\hat{k_1}(z_1) + \hat{k_2}(z_2)]$, $(z_1, z_2) \in \mathbf{R}^2 + i\mathbf{R}^2_+$. *Then, there exists the unique solution* $\varphi \in L^1(\mathbf{R}^2_+)$ *of (5).*

3. *Let* $F = c\delta(t,s) + k_1(t)\delta(s) + k_2(s)\delta(t) + k(t,s)$ *where* $k_1(t) \in LG'_0(\mathbf{R}^1_+)$, $k_2(s) \in LG'_0(\mathbf{R}^1_+)$, *and* P *be a polynomial. Then, the necessary and sufficient condition for the existence of the solution of the equation* $(P(\delta) + F) * \varphi = h$ *(in particular, equation (5)) in* $LG'_0(\mathbf{R}^2_+)$, *for every* $h \in LG'_0(\mathbf{R}^2_+)$ *is that there exists* $\alpha, \beta \in \mathbf{R}, C > 0$ *such that*

$$\left| \frac{1}{P(-iz_1, -iz_2) + \hat{F}} \right| \leq C \frac{(1 + |z_1| + |z_2|)^\alpha}{(\min |y_1|, |y_2|)^\beta},$$

$z = x + iy$, $(z_1, z_2) \in \mathbf{R}^2 + i\mathbf{R}^2_+$. *Then, the solution of this equation (particularly, equation (5)) is unique.*

4. *The equation*

$$\varphi * \varphi = h,$$

$h \in LG'_0(\mathbf{R}^2_+)$ *has the unique solution in* $LG'_0(\mathbf{R}^2_+)$ *if and only if* $\sqrt{\hat{h}(z_1, z_2)}, (z_1, z_2) \in \mathbf{R}^2 + i\mathbf{R}^2_+$ *is analytic in* $\mathbf{R}^2 + i\mathbf{R}^2_+$.

Remark. In the case when F is of the form 2, then this equation has an integral analogue

$$(6)\ c\varphi(t,s) + \int_0^\infty k_1(t-\tau)\varphi(\tau,s)d\tau + \int_0^\infty k_2(s-\sigma)\varphi(t,\sigma)d\sigma = f(t,s),$$

$c \in \mathbf{C}$, $k_1, k_2 \in L^1(\mathbf{R}_+)$, (see [1]). This holds because of

$$k_1 * F(t,s) = (k_1(\tau) \otimes \delta(\sigma) * F(\tau,\sigma))(t,s) = \int_0^t k_1(\tau)F(t-\tau,s)d\tau.$$

The similar holds in case 3, when the integral analogue is given by

$$(7)\, f(t,s) = c\varphi(t,s) + \int_0^\infty k_1(t-\tau)\varphi(\tau,s)d\tau + \int_0^\infty k_2(s-\sigma)\varphi(t,\sigma)d\sigma +$$

$$\int_0^\infty \int_0^\infty k(t-\tau, s-\sigma)\varphi(\tau,\sigma)d\tau d\sigma.$$

Proof. 1. All holomorphic functions of the form

$$f(z_1,z_2) = c + \int\int_{\mathbf{R}_+^2} \varphi(t,s)e^{i(z_1t+z_2s)}dtds, \tag{8}$$

$$(z_1,z_2) \in \mathbf{R}^2 + i\mathbf{R}_+^2, c \in \mathbf{C}, \varphi \in L^1(\mathbf{R}_+^2)$$

with $\int\int_{\mathbf{R}_+^2} \varphi(t,s)e^{i(z_1t+z_2s)} \neq -c, (z_1,z_2) \in (\mathbf{R}^2 + i\mathbf{R}_+^2) \bigcup \dot{\mathbf{R}}^2$ constitute the Wiener algebra $W(\mathbf{R}_+^2)$.

Let $f(z_1,z_2) \neq 0, (z_1,z_2) \in (\mathbf{R}^2 + i\mathbf{R}_+^2) \bigcup \dot{\mathbf{R}}^2$, be of the form (8). Then, by [10] (Ch. II, & 13.4), there exists $G \in V_+^2$ where $\mathcal{L}(V_+^2) = W(\mathbf{R}_+^2)$, such that

$$[c\delta(t,s) + k(t,s)] * G(t,s) = \delta(t,s).$$

G is the fundamental solution. This implies that

$$\varphi(t,s) = G(t,s) * h(t,s), \varphi(t,s) \in L^1(\mathbf{R}_+^2).$$

2. It follows from Oscher's theorem [1], Ch. 9.51, p. 423.

3. The assertion directly follows from the general results given in [11], (Ch. I, & 7, p. 50).

4. Using the same results of [11] as in the previous case we obtain that the assumption of the assertion implies that $\sqrt{h} = \mathcal{L}(\varphi)$ for some $\varphi \in LG_0'$. If $\varphi \in LG_0'(\mathbf{R}_+^2)$, satisfies $\varphi * \varphi = h$, then $\hat{h} = \hat{\varphi}^2$, and $\hat{\varphi}$ is analytic with $\hat{\varphi} = \sqrt{\hat{h}}$.

Laguerre series solution of these integral equations

We shall give the Laguerre series solution of equations (5) with F given in Theorem 1 by 1, 2, 3 and 4, respectively. Let $\varphi(t,s) = \sum_{nm} x_{nm} l_n(t) l_m(s)$, $k(t,s) = \sum_{nm} a_{nm} l_n(t) l_m(s)$.

Then 1 has the form

$$\sum_{(n,m)\in \mathbf{N}_0^2} [cx_{nm}+\sum_{p+q=n}\sum_{r+s=m} a_{pr}x_{qs}-\sum_{p+q=n-1}\sum_{r+s=m} a_{pr}x_{qs}-\sum_{p+q=n}\sum_{r+s=m-1} a_{pr}x_{qs}$$

$$+\sum_{p+q=n-1}\sum_{r+s=m-1} a_{pr}x_{qs}]l_n(t)l_m(s) = \sum_{(n,m)\in \mathbf{N}_0^2} d_{nm}l_n(t)l_m(s).$$

So we obtain the following system of equations:

$$cx_{nm} + \sum_{p+q=n}\sum_{r+s=m} a_{pr}x_{qs} - \sum_{p+q=n-1}\sum_{r+s=m} a_{pr}x_{qs} - \sum_{p+q=n}\sum_{r+s=m-1}$$

$$a_{pr}x_{qs} + \sum_{p+q=n-1}\sum_{r+s=m-1} a_{pr}x_{qs} = d_{nm}, \quad n,m\in \mathbf{N}_0.$$

The solution is given by:

$$x_{nm} = \frac{1}{(c+a_{00})}[d_{nm} + \sum_{i+j=0}^{n+m-1} x_{ij}\sum_{p=i}^{n}\sum_{q=j}^{m}(-1)^{i+j}a_{pq}], n,m\in \mathbf{N}_0.$$

Because of $\delta(t,s) = \sum_{(n,m)\in \mathbf{N}_0^2} l_n(t)l_m(s)$ (see Appendix) and $\delta(x) = \sum_{n=0}^{\infty} l_n(x)$ we have the convolution analogue for the equation (6):

$$[c\sum_{(n,m)\in \mathbf{N}_0^2} l_n(t)l_m(s) + \sum_{n=0}^{\infty}(\sum_{p+q=n}(a_p+b_p)l_p(t)l_q(s))] * \phi(t,s) = f(t,s).$$

Applying the algorithm for solving convolution equations of the form (5) given by (4) we obtain the solution of equation (6).

Finally, we consider equation (7). We shall solve this equation by utilizing the two previous ones. The left-hand side of (5) is given by

$$F(t,s) = c\delta(t,s) + k_1(t)\delta(s) + k_2(s)\delta(t) + k(t,s),$$

and it has the Laguerre series form

$$F(t,s) = \sum_{(n,m)\in \mathbf{N}_0^2} (c+r_{nm})l_n(t)l_m(s) + \sum_{(n,m)\in \mathbf{N}_0^2} (\sum_{p+q=n}(a_p+b_q)l_p(t)l_q(s)).$$

where $k(t,s) = \sum_{(n,m)\in \mathbf{N}_0^2} r_{nm}l_n(t)l_m(s)$. The solution is given by (4).

Laguerre series solution in the non-linear case.

We have the Laguerre series solution in the case $\varphi * \varphi = h$ in the form of the system of equations:

$$\sum_{p+q=n}\sum_{r+s=m} x_{pq}x_{rs} - \sum_{p+q=n-1}\sum_{r+s=m} x_{pr}x_{qs} -$$

$$\sum_{p+q=n}\sum_{r+s=m-1} x_{pr}x_{qs} + \sum_{p+q=n-1}\sum_{r+s=m-1} x_{pr}x_{qs} = c_{nm}, (n,m) \in \mathbf{N}_0^2,$$

where

$$x_{nm} = 1/x_{00}[\sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} - \sum_{p+q=n}\sum_{r+s=m} x_{pr}x_{qs} + x_{00}x_{nm}], (n,m) \in \mathbf{N}_0^2,$$

with $\varphi(t,s) = \sum_{(n,m)\in\mathbf{N}_0^2} x_{nm} l_n(t) l_m(s)$.

7. Generalized error estimate

Lemma 1. *Let $a_{ij} \geq 0$, $i,j \in \mathbf{N}_0$. Then,*

1.

$$\sum_{n=l}^{\infty}\sum_{m=r}^{\infty} \frac{1}{(nm)^2}(\sum_{i=l}^{n}\sum_{j=r}^{m} |a_{ij}|)^2 \leq 4\sum_{n=l}^{\infty}\sum_{m=r}^{\infty} |a_{nm}|^2.$$

2. If

$$\sum_{q=1}^{\infty}\sum_{s=1}^{\infty} \frac{a_{qs}^2}{q^{2r-3}s^{2k-3}} < \infty,$$

then

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{1}{i^{2r}j^{2k}}(\sum_{q=1}^{i}\sum_{s=1}^{j} a_{qs})^2 \leq \frac{4}{((n+1)(m+1))^2}$$

$$((\sum_{q\leq n}\sum_{s=1}^{j} + \sum_{q=1}^{i}\sum_{s\leq m})\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2 + 8\sum_{q=n+1}^{\infty}\sum_{s=m+1}^{\infty} \frac{a_{qs}^2}{q^{2r-3}s^{2k-3}} \to 0,$$

as $n,m \to \infty$.

Proof. 1. Recall the Hardy-Landay inequality for two-dimensional sequences: Let $p > 0$, $a_{ij} \geq 0, i, j \in \mathbf{N}$. Then

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{(nm)^p}(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|)^p \leq (\frac{p}{p-1})^p\sum_{n=1}^{N}\sum_{m=1}^{M}|a_{nm}|^2.$$

Setting $p = 2$ and $a_{ij} = 0$ if $i \leq l$ or $j \leq r$, we obtain

$$\sum_{n=l}^{N}\sum_{m=r}^{M}\frac{1}{(nm)^2}(\sum_{i=l}^{n}\sum_{j=r}^{m}|a_{ij}|)^2 \leq 4\sum_{n=l}^{N}\sum_{m=r}^{M}|a_{nm}|^2.$$

By letting $N, M \to \infty$ we obtain 1.

2. We have

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}\frac{1}{i^{2r}j^{2k}}(\sum_{q=1}^{i}\sum_{s=1}^{j}a_{qs})^2 \leq \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}\frac{4}{i^{2r}j^{2k}}[((\sum_{q\leq n}\sum_{s=1}^{j}+\sum_{q=1}^{i}\sum_{s\leq m})$$

$$a_{qs})^2 + (\sum_{q=n+1}^{i}\sum_{s=m+1}^{j}a_{qs})^2] \leq \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}$$

$$\frac{4}{i^3j^3}[((\sum_{q\leq n}\sum_{s=1}^{j}+\sum_{q=1}^{i}\sum_{s\leq m})\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2$$

$$+(\sum_{q=n+1}^{\infty}\sum_{s=m+1}^{\infty}\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2] \leq \frac{4}{((n+1)(m+1))^2}$$

$$((\sum_{q\leq n}\sum_{s=1}^{m}+\sum_{q=1}^{n}\sum_{s\leq m})\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2+$$

$$4\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}\frac{1}{i^2j^2}(\sum_{q=n+1}^{i}\sum_{s=m+1}^{j}\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2.$$

By Lemma 1 part 1, we obtain that this estimate is

$$\leq \frac{4}{((n+1)(m+1))^2}(\sum_{q\leq n}\sum_{s=1}^{j}+\sum_{q=1}^{i}\sum_{s\leq m})(\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2$$

$$+8\sum_{q=n+1}^{\infty}\sum_{s=m+1}^{\infty}(\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}})^2.$$

Because of

$$\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}} \to 0, \text{ as } q,s\to\infty,$$

we have

$$\frac{1}{nm}(\sum_{q\le n}\sum_{s=1}^{j}+\sum_{q=1}^{i}\sum_{s\le m})\frac{a_{qs}}{q^{r-3/2}s^{k-3/2}} \to 0,\ n,m\to\infty.$$

□

Recall, $\mathcal{L}(V_+^2)=W(\mathbf{R}_+^2)$, where W is the Wiener algebra.

Theorem 2.

1. *If $F=c\delta+\varphi\in V_+^2$, then the solution of $F*G=\delta$ is of the form $G=(1/c)\delta+\varphi_1,\varphi_1\in L^1(\mathbf{R}_+^2)$. Let $G=\sum_{nm}a_{nm}l_nl_m$. Then,*

$$|a_{nm}|\le C, n,m\in\mathbf{N}_0, \text{ for some } C>0.$$

2. *Let $h\in LG'_{0(r-3/2,k-3/2)}$, $r,k>3/2$. Then, the error estimate for the solution $g=G*h$ of $F*g=h$ is given by*

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}\frac{|x_{ij}|^2}{i^{2r}j^{2k}}\le C[\frac{4}{((n+1)(m+1))^2}$$

$$((\sum_{q\le n}\sum_{s=1}^{j}+\sum_{q=1}^{i}\sum_{s\le m})\frac{|h_{q,s}-h_{q-1,s}-h_{q,s-1}+h_{q-1,s-1}|}{\dot{q}^{r-3/2}\dot{s}^{k-3/2}})^2$$

$$+8\sum_{q=n+1}^{\infty}\sum_{s=m+1}^{\infty}\frac{|h_{q,s}-h_{q-1,s}-h_{q,s-1}+h_{q-1,s-1}|^2}{\dot{q}^{2r-3}\dot{s}^{2k-3}})],$$

where $h=\sum_{nm}h_{nm}l_nl_m$, $g=\sum_{nm}x_{nm}l_nl_m$ and C is from 1., $h_{v,u}=0$, when $v<0$ or $u<0$, and $\dot{q},\dot{s}=\begin{cases}1, & q,s=0\\ q,s & q,s>0.\end{cases}$

Proof. 1. The first part of 1. is from [10], Ch. II, & 13, 4. Because $\delta(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} l_n(x)l_m(y)$, it is enough to prove that there is a constant $C > 0$ such that $| < l_n l_m, \varphi_1 > | \le C$, $n, m \in \mathbf{N}_0$. This follows from

$$|e^{-(x+y)/2}L_n(x)L_m(y)| \le C,\ (x,y) \in \bar{\mathbf{R}}_+^2, (n,m) \in \mathbf{N}_0^2,$$

C is a suitable constant (see [3]).

2. The approximate solution of $F * g = h$, $g_{nm} = \sum_{i=0}^n \sum_{j=0}^m x_{ij} l_i l_j$ is given by $g_{nm} = G_{nm} * h_{nm} = G * h_{nm} = G_{nm} * h$, $n, m \in \mathbf{N}$. Because $g - g_{nm}$ is a tempered distribution over the quarter plane we must estimate $\sum_{i=n+1}^\infty \sum_{j=m+1}^\infty |x_{ij}|^2/(i^{2r}j^{2k})$. Because of

$$x_{ij} = \sum_{p+q=i} \sum_{r+s=j} a_{pr}(h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}),\ (i,j) \in \mathbf{N}_0^2$$

and from 1. it follows

$$\sum_{i=n+1}^\infty \sum_{j=m+1}^\infty \frac{|x_{ij}|^2}{i^{2r}j^{2k_0}} \le C \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty \frac{1}{i^3j^3}$$

$$\left(\frac{(\sum_{q\le n}\sum_{s=1}^j + \sum_{q=1}^i \sum_{s\le m})|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|}{i^{r-3/2}j^{k-3/2}}\right)^2$$

$$\le C \sum_{i=n+1}^\infty \sum_{j=m+1}^\infty \frac{1}{((i+1)(j+1))^3}\Bigg(\Big(\sum_{q\le n}\sum_{s=1}^{j} + \sum_{q=1}^{i}\sum_{s\le m}\Big)$$

$$\frac{|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|}{\dot{q}^{r-3/2}\dot{s}^{k-3/2}}\Bigg)^2.$$

Now, applying Lemma 1 part 2 we obtain

$$\le C\Bigg[\frac{4}{((n+1)(m+1))^2}\Bigg(\Big(\sum_{q\le n}\sum_{s=1}^{j} + \sum_{q=1}^{i}\sum_{s\le m}\Big)\frac{|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|}{\dot{q}^{r-3/2}\dot{s}^{k-3/2}}\Bigg)^2$$

$$+8 \sum_{q=n+1}^\infty \sum_{s=m+1}^\infty \frac{|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|^2}{\dot{q}^{r-3/2}\dot{s}^{k-3/2}}\Bigg]$$

and the proof follows.

Remark. If h has better regular properties we can obtain a sharper error estimate. Consider case 1 in Theorem 1. Since $g = (1/c)h + \varphi_1 * h$, and $g_{nm} = (1/c)h_{nm} + \varphi_1 * h_{nm}$, we have

$$g - g_{nm} = (1/c)(h - h_{nm}) + \varphi_1 * (h - h_{nm}).$$

Theorem 3. *If $h \in L^\infty(\mathbf{R}^2_+)$ and $h_{nm} \overset{L^\infty}{\to} h$, $n,m \to \infty$, then there exists $C > 0$ such that*

$$||g - g_{nm}||_{L^\infty} \leq C||h - h_{nm}||_{L^\infty}.$$

Proof. Because $\varphi_1 \in L^1(\mathbf{R}^2_+)$ the proof follows from

$$||\varphi_1 * (h - h_{nm})||_{L^\infty} \leq C||\varphi||_{L^1}||(h - h_{nm})||_{L^\infty}.$$

□

Let $F \in LG'_0(\mathbf{R}^2_+)$, and $F = \sum_{(n,m)\in \mathbf{N}^2_0} a_{nm} l_n l_m$ in $F * G = \delta$. Then, there are $C \geq 0$, $t > 0$, and $l > 0$ such that

$$|a_{nm}| \leq C n^t m^l, \ (n,m) \in \mathbf{N}^2_0. \tag{9}$$

The following estimate holds:

Theorem 4. *Let $h \in LG'_{0(r-3/2,k-3/2)}(\mathbf{R}^2_+)$, $r,k > 3/2$ and r,l be from (9). Then, for the solution G of $F * G = h$ we obtain the following estimate:*

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{|x_{ij}|^2}{i^{2(r+t)}j^{2(k+l)}} \leq$$

$$C[\frac{4}{((n+1)(m+1))^2}((\sum_{q\leq n}\sum_{s=1}^{j} + \sum_{q=1}^{i}\sum_{s\leq m})\frac{|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|}{\dot{q}^{r-3/2}\dot{s}^{k-3/2}})^2$$

$$+8\sum_{q=n+1}^{\infty}\sum_{s=m+1}^{\infty} \frac{|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|^2}{\dot{q}^{2(r-3)}\dot{s}^{2(k-3)}}].$$

Proof. From (9) we obtain

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{|x_{ij}|^2}{i^{2(r+t)}j^{2(k+l)}} \leq \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{i^{2t}j^{2l}}{i^{2(r+t)}j^{2(k+l)}}$$

$$|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|^2 \leq C\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty}\frac{1}{i^3 j^3}$$

$$(\frac{(\sum_{q\leq n}\sum_{s=1}^{j} + \sum_{q=1}^{i}\sum_{s\leq m})|h_{q,s} - h_{q-1,s} - h_{q,s-1} + h_{q-1,s-1}|}{\dot{i}^{r-3/2}\dot{j}^{k-3/2}})^2.$$

Repeating the proof of Theorem 2, part 2, we obtain the error estimate in Theorem 4.

8. Solving partial differential equations

We shall give some simple test examples of partial differential equations which can be solved by our method.

Consider the equation

$$-\Delta u + u = f \ \text{ in } \ \mathbf{R}^2_+.$$

It means $[-(\delta_{xx} + \delta_{yy}) + \delta] * u = f$. From Appendix we have

$$\delta_{xx}(x,y) = \sum_{(n,m)\in \mathbf{N}_0^2} [n(n-1)/2 + n + 1/4] l_n(x) l_m(y) \tag{10}$$

and a similar formula for δ_{yy}.

The existence and uniqueness of the solution follow from Theorem 1 part 3. The solution belongs to the space $LG'(\mathbf{R}^2_+)$, and it is given by $a_{nm} = -(1/2)(n^2 + m^2 + n + m - 1)$.

The following example is not contained in Theorem 1, but it has solution in $LG'_e(\mathbf{R}^2_+)$ according to Proposition 4, since $x_{00} \neq 0$.

The convection-diffusion phenomena governed by the equation of the type

$$\frac{\partial u}{\partial t} = \epsilon \frac{\partial u^2}{\partial x^2} - k \frac{\partial u}{\partial x} + g, \ g \in LG'_e$$

on the strip $(x,t) \in \mathbf{R}^2_+$, has the convolution form

$$(\epsilon \delta_{xx} - k\delta_x - \delta_t) * u = g.$$

By using $\delta_t = \sum_{nm}(m + 1/2) l_n(x) l_m(t)$ and (10) we have that $a_{00} = \epsilon/4 - k/2 - 1/2 \neq 0$ and the unique solution is given by

$$x_{nm} = \frac{1}{(\epsilon/4 - k/2 - 1/2)} \{ \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij}$$

$$- \sum_{p+r=n} \sum_{q+s=m} a_{pr} x_{qs} + a_{00} x_{nm} \}, \ (n,m) \in \mathbf{N}_0^2,$$

where

$$a_{nm} = \epsilon(n^2/2 + n/2 + 1/4) - k(n + 1/2) - (m + 1/2), \ (n,m) \in \mathbf{N}_0^2,$$

and c_{nm}, $(n,m) \in \mathbf{N}_0^2$ are the Laguerre coefficients of g.

For hyperbolic integro-differential equation

$$u_{tt} - u_{xx} - bu_x - cu = \int_0^t k(\tau)u(x,t-\tau)d\tau + \delta(x,t)$$

the evaluation of corresponding terms in the convolution form

$$[\delta_{tt}(t,x) - \delta_{xx}(t,x) - b\delta_x(t,x) - c\delta(t,x) - k(t)\delta(x)] * u = \delta(t,s)$$

gives that $a_{00} \neq 0$ and thus, it is uniquely solvable in LG'_e by our approximation method.

9. Appendix

Evaluation of some distributions in the Laguerre series

Partial derivatives of an $f \in LG'_0(\mathbf{R}_+^2)$.

Since $f'(x) = \sum_{i=0}^{n-1}(b_i + 1/2b_n)l_n(x)$ if $f \in LG'_0(\mathbf{R}_+^1)$, then if $f(x,y) \in LG'_0(\mathbf{R}_+^2)$ its derivatives on x are given by

$$\frac{\partial}{\partial x} f(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} (\sum_{i=0}^{n-1} a_{im} + 1/2a_{nm})l_n(x)l_m(y).$$

Analogously,

$$\frac{\partial}{\partial y} f(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} (\sum_{j=0}^{m-1} a_{nj} + 1/2a_{nm})l_n(x)l_m(y).$$

Differentiating repeadetly we obtain

$$\frac{\partial^2}{\partial^2 y} f(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} [\sum_{i=0}^{m-1}\sum_{j=0}^{i-1} a_{ni} + \sum_{i=0}^{m-1} a_{ni} + 1/4a_{nm}]l_n(x)l_m(y).$$

Examples.

$$J_0(\sqrt{xt}) = \sum_{n=0}^{\infty} 2(-1)^n l_n(x)l_n(t),$$

where J_0 is the Bessel function of the first kind and zero order.

$$H(x,y) = \begin{cases} 1 & x,y>0 \\ 0 & \text{otherwise} \end{cases} = H(x)H(y) =$$

$$4 \sum_{(n,m)\in\mathbf{N}_0^2} (-1)^{n+m} l_n(x) l_m(y),\ (n,m) \in \mathbf{N}_0.$$

$$H(x,y)e^{-sx-ry} = \sum_{(n,m)\in\mathbf{N}_0^2} 4(-1)^{m+n} \frac{(s-1/2)^n (r-1/2)^m}{(s+1/2)^{n+1}(r+1/2)^{m+1}} l_n(x) l_m(y),$$

$$\delta(x-a, y-b) = \sum_{(n,m)\in\mathbf{N}_0^2} l_n(a) l_m(b) l_n(x) l_m(y),\quad a,b>0,$$

$$\delta_x(x,y) = \frac{\partial}{\partial x}\delta(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} (n+1/2) l_n(x) l_m(y),$$

$$\delta_{xx}(x,y) = \sum_{(n,m)\in\mathbf{N}_0^2} (n(n-1)/2 + n + 1/4) l_n(x) l_m(y).$$

Because of

$$H(x-a) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases} = \sum_{n=0}^{\infty}[\sum_{j=0}^{n-1} 4(-1)^{n-j} l_j(a) + 2l_n(a)] l_n(x),$$

(see [6]) we have

$$H(x-a, y-b) = H(x-a)H(y-b) = \sum_{(n,m)\in\mathbf{N}_0^2} [\sum_{j=0}^{n-1} 4(-1)^{n-j} l_j(a) + l_n(a)]$$

$$[\sum_{j=0}^{m-1} 4(-1)^{m-j} l_j(b) + l_m(b)] l_n(x) l_m(y).$$

Since

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0 & x < 0 \end{cases} = 2^{\alpha+1}\alpha! \sum_{n=0}^{\infty}[\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\alpha+k}{\alpha} 2^k] l_n(x)$$

(see [6]) we obtain

$$x_+^\alpha y_+^\beta = \sum_{(n,m)\in \mathbf{N}_0^2} 2^{\alpha+\beta+2}(\alpha!)^2[\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha+k}{\alpha}2^k]$$

$$[\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{\beta+k}{\beta}2^k]l_n(x)l_m(y).$$

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