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THE SPACE OF FUNCTIONS WITH A LIMIT AT EACH POINT

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Abstract

We consider the set $\Lambda(X,Y)$ of functions $f : X \to Y$ which have a limit at each point of X. If X is a compact space, $\Lambda(X,Y)$ is a Banach algebra. The representation of bounded linear functionals on $\Lambda(X,Y)$ is given.

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1. Introduction

In all further considerations, $(X, \mathcal{O}_X)$ will denote a topological space without isolated points, while $(Y, \mathcal{O}_Y)$ will be a Hausdorff space. These conditions will ensure the uniqueness of the limit $\lim_x f$, where $f : X \to Y$ and $x \in X$. We consider the set

$$\Lambda(X,Y) = \{f \in Y^X : \forall x \in X \quad \exists y_x \in Y \quad y_x = \lim_x f\}$$

which is larger than $C(X,Y)$. This is shown by the following well-known examples.

Example 1.1. Let $Q = \{q_n : n \in N\}$ be the space of rationals and let $f : Q \to R$ be defined by $f(q_n) = 1/n$, $n \in N$. Then, for each $q \in Q$ we have $\lim_q f = 0 \neq f(q)$. So, f is discontinuous at each point although $f \in \Lambda(Q, R)$.

Example 1.2. The function $f : R \to R$ given by:

$$f(x) = \begin{cases} 1/n & \text{if} \quad x = q_n,\ n \in N; \\ 0 & \text{if} \quad x \in R \setminus Q \end{cases}$$

is continuous at each point of $R \setminus Q$ and it is discontinuous at each rational point. Moreover, $\lim_x f = 0$ for all $x \in R$, thus $f \in \Lambda(R, R)$.

Theorem 1.1. *Suppose $(X, \mathcal{O}_X)$ is a T_1–space, $(Y, \mathcal{O}_Y)$ is a regular space, $D \subset X$, $D' = X$ and $f : D \to Y$ where*

$$\forall x \in X \;\; \exists y_x \in Y \;\; y_x = \lim_{t \to x}^{t \in D} f(t).$$

Then, the function $F : X \to Y$ defined by $F(x) = y_x$, for all $x \in X$, is continuous.

Proof. a) By the assumption for each $x \in X$ there is $y_x \in Y$ such that

$$\forall W \in \mathcal{U}(y_x) \;\; \exists V \in \mathcal{U}(x) \; f(V \setminus \{x\} \cap D) \subset W. \tag{1}$$

Let us prove the continuity of F at arbitrary point $x \in X$. Let $U \in \mathcal{U}(y_x)$. Since Y is a regular space there is $W \in \mathcal{U}(y_x)$ such that $\overline{W} \subset U$, and by (1), there exists $V \in \mathcal{U}(x)$ satisfying

$$d \in V \setminus \{x\} \cap D \Rightarrow f(d) \in W. \tag{2}$$

Suppose that $z \in V$ and $F(z) = y_z \notin \overline{W}$. Then $y_z \in Y \setminus \overline{W} \in \mathcal{U}(y_z)$ and (1) gives $G \in \mathcal{U}(z)$ such that

$$d \in G \setminus \{z\} \cap D \Rightarrow f(d) \in Y \setminus \overline{W}. \tag{3}$$

$z = x$ would imply $y_z = y_x \in W \cap Y \setminus \overline{W} = \emptyset$, therefore $z \neq x$ and $z \in G \cap V \setminus \{x\} \in \mathcal{U}(z)$. Since $D' = X$ there is a $d \in D \cap G \cap V \setminus \{x, z\}$ and from (2) and (3) it follows that $f(d) \in W \cap Y \setminus \overline{W} = \emptyset$ which is impossible. Thus $F(V) \subset \overline{W} \subset U$ and F is continuous at the point x. □

As a special case of the previous theorem, for $D = X$, we have

Corollary 1.1. *If $(X, \mathcal{O}_X)$ is a T_1–space, $(Y, \mathcal{O}_Y)$ is a regular space and $f \in \Lambda(X,Y)$, then the function $F : X \to Y$ given by $F(x) = y_x$ is continuous.* □

Example 1.3. If f is the function from the Example 1.1, then $F(x) = 0 \neq f(x)$, for all $x \in X$.

Theorem 1.2. *Let (Y, d) be a metric space and $f \in \Lambda(X,Y)$. Then*

a) For all $r > 0$, the set $\Delta_r = \{x \in X \mid d(y_x, f(x)) > r\}$ has no accumulation points in X.

b) Δ_r is a closed, discrete subspace of X and $\mid \Delta(f) \mid \leq e(X)$, where $\Delta(f) = \{x \in X : f$ is discontinuous at $x\}$ and $e(X) = \sup\{\mid D \mid : D \subset X$ is closed and discrete $\}$.

c) If X is a separable metrizable space, then $\mid \Delta(f) \mid \leq \omega$.

d) For all $r > 0, \Delta_r$ is a nowhere dense set.

e) If the space X is metrizable with a complete metric, then the mapping f is continuous on a set of the second category.

f) If $(X, \mathcal{O})$ is a compact space, then Δ_r is a finite set (for each $r > 0$) and f is a bounded function.

Proof. a) Suppose that for some $r > 0$ we have $\Delta'_r \neq \emptyset$, i.e.

$$\exists x \in X \;\; \forall U \in \mathcal{U}(x) \;\; \exists x_U \in U \setminus \{x\} \cap \Delta_r. \tag{4}$$

Then, there is a net $< x_U \mid U \in \mathcal{U}(x) > \to x$. By the continuity of F from Corollary 1.1 we have $< y_{x_U} \mid U \in \mathcal{U}(x) > \to y_x$, hence there is $W \in \mathcal{U}(x)$ such that

$$\forall G \in \mathcal{U}(x)(G \subset W \Rightarrow d(y_{x_G}, y_x) < r/2). \tag{5}$$

Since $\lim_x f = y_x$, there is $V \in \mathcal{U}(x)$ satisfying

$$\forall t \in V \setminus \{x\} \;\; d(f(t), y_x) < r/2. \tag{6}$$

If $O \in \mathcal{U}(x)$ and $O \subset V \cap W$, then $x_O \in W \cap V \setminus \{x\}$ and by (5) and (6)

$$d(y_{x_O}, f(x_O)) \leq d(y_{x_O}, y_x) + d(y_x, f(x_O)) < r.$$

A contradiction to $x_O \in \Delta_r$.

b) From (4), for each $x \in \Delta_r$ there is $U \in \mathcal{U}(x)$ such that $U \cap \Delta_r = \{x\}$ so, the discreteness is verified. Since $\overline{\Delta_r} = \Delta_r \cup \Delta_r' = \Delta_r$, Δ_r is closed. Finally, for all $n \in N$ we have $| \Delta_{1/n} | \leq e(X)$, hence $| \Delta(f) | = | \bigcup_{n \in N} \Delta_{1/n} | \leq \omega e(X) = e(X)$.

c) In a separable metric space we have $e(X) = d(X) = \omega$.

d) Suppose that $z \in \mathrm{int}\overline{\Delta_r} = \mathrm{int}\Delta_r$. Choose $U \in \mathcal{U}(z)$ such that $U \cap \Delta_r = \{z\}$. Then $U \cap \mathrm{int}\Delta_r = \{z\}$, which is impossible because X has no isolated points.

e) Follows from $(d), \Delta(f) = \bigcup_{n \in N} \Delta_{1/n}$ and the Baire Category Theorem.

f) Δ_r is finite because of (a). Let $\Delta_1 = \{x_1, ..., x_k\}$ and let F be the function from Corollary 1.1. Then

$$\forall x \in X \setminus \Delta_1 \quad d(f(x), F(x)) \leq 1$$

and for all $x, y \in X \setminus \Delta_1$ we have

$$d(f(x), f(y)) \leq d(f(x), F(x)) + d(F(x), F(y)) + d(F(y), f(y)) \leq 2 + \rho(F(X)).$$

Now, $F(X)$ is a compact set in Y, thus it must be bounded. So, $\rho(f(X \setminus \Delta_1)) \leq 2 + \rho(F(X)) < \infty$ and since $f(\Delta_1)$ is a bounded set we have $\rho(f(X)) < \infty$. □

2. $\Lambda(X, R)$ as a Banach algebra

The space of all bounded real-valued functions $B(X, R)$ with the norm

$$\|f\| = \sup_{x \in X} | f(x) |$$

is a commutative Banach algebra with the unit.

Theorem 2.1. *If $(X, \mathcal{O})$ is a compact, then $\Lambda(X, R)$ is a Banach subalgebra of $B(X, R)$.*

Proof. By Theorem 1.2(f) we have $\Lambda(X, R) \subset B(X, R)$. Also, if $\lim_x f$ and $\lim_x g$ exist, then $\lim_x(\alpha f + \beta g)$ and $\lim_x fg$ exist, hence $\Lambda(X, R)$ is a subalgebra of $B(X, R)$.

Let us prove that $\Lambda(X,R)$ is a closed subset of $B(X,R)$. Suppose $< f_n \mid n \in N > \to f$, where $f_n \in \Lambda(X,R)$, $n \in N$, i.e.

$$(7) \qquad \forall \epsilon > 0\ \exists n_0 \in N\ \forall n \geq n_0 \forall x \in X \mid f_n(x) - f(x) \mid < \epsilon.$$

A convergent sequence is bounded, thus there is $M > 0$ such that

$$(8) \qquad \forall n \in N\ \exists x \in I \mid f_n(x) \mid < M.$$

Let $x_0 \in X$ and $y^n_{x_0} = \lim_{x_0} f_n, n \in N$. Since $\mid \cdot \mid$ is a continuous function, from (8) we have $\lim_{x_0} \mid f_n \mid = \mid y^n_{x_0} \mid \leq M$. So, $< y^n_{x_0} \mid n \in N >$ is a bounded real sequence.

Suppose that α and β are the accumulation points of the sequence $< y^n_{x_0} \mid n \in N >$. By (7), for $\epsilon = \mid \alpha - \beta \mid /6$ there is $n_0 \in N$ satisfying

$$(9) \qquad \forall n \geq n_0 \forall x \in X \mid f_n(x) - f(x) \mid < \epsilon.$$

Choose $n_1, n_2 \geq n_0$ such that $\mid y^{n_1}_{x_0} - \alpha \mid$, $\mid y^{n_2}_{x_0} - \beta \mid < \epsilon$. Since $y^{n_i}_{x_0} = \lim_{x_0} f_{n_i}$, $i = 1,2$, there is $U \in \mathcal{U}(x_0)$ such that for each $x \in U \setminus \{x_0\}$ we have $\mid f_{n_i}(x) - y^{n_i}_{x_0} \mid < \epsilon$, $i = 1,2$. Choose such $x \in U \setminus \{x_0\}$ (this is possible because x_0 is not an isolated point). Now, we have: $\mid \alpha - \beta \mid \leq \mid \alpha - y^{n_1}_{x_0} \mid + \mid y^{n_1}_{x_0} - f_{n_1}(x) \mid + \mid f_{n_1}(x) - f(x) \mid + \mid f(x) - f_{n_2}(x) \mid + \mid f_{n_2}(x) - y^{n_2}_{x_0} \mid + \mid y^{n_2}_{x_0} - \beta \mid < 6\epsilon = \mid \alpha - \beta \mid$, a contradiction! The sequence $< y^n_{x_0} \mid n \in N >$ converges.

Let $< y^n_{x_0} \mid n \in N > \to y_{x_0}$. We will prove that $\lim_{x_0} f = y_{x_0}$, i.e.

$$(10) \qquad \forall \epsilon' > 0\ \exists U \in \mathcal{U}(x_0) \forall x \in U \setminus \{x_0\} \mid f(x) - y_{x_0} \mid < \epsilon'.$$

Given $\epsilon' > 0$, for $\epsilon = \epsilon'/3$, there is $n_0 \in N$ satisfying (9). Let $m \in N$ be such that $m \geq n_0$ and $\mid y^m_{x_0} - y_{x_0} \mid < \epsilon$. Then, by (9)

$$\forall x \in X \mid f_m(x) - f(x) \mid < \epsilon.$$

Since $y^m_{x_0} = \lim_{x_0} f_m$, there exists a neighbourhood $U \in \mathcal{U}(x_0)$ such that

$$\forall x \in U \setminus \{x_0\} \mid f_m(x) - y^m_{x_0} \mid < \epsilon.$$

According to the previous inequalities, for each $x \in U \setminus \{x_0\}$ we have

$$\mid f(x) - y_{x_0} \mid \leq \mid f(x) - f_m(x) \mid + \mid f_m(x) - y^m_{x_0} \mid + \mid y^m_{x_0} - y_{x_0} \mid < 3\epsilon = \epsilon',$$

and (10) is proved. Thus, for each $x_0 \in X$ there exists $\lim_{x_0} f$, hence $f \in \Lambda(X, R)$. □

Remark 2.1. Clearly, the Banach algebra $C(X,R)$ is a Banach subalgebra of $\Lambda(X,R)$. Moreover, for each $F \in C(X,R)$ and $r > 0$ there is $f \in B(F,r) \setminus C(X,R)$ given by

$$f(x) = \begin{cases} F(x) & \text{for} \quad x \neq x_0 \\ F(x) + r/2 & \text{for} \quad x = x_0 \end{cases}$$

where $x_0 \in X$. ($B(F,r)$ is an open ball). So, $C(X,R)$ is a nowhere dense subspace of $\Lambda(X,R)$.

For a function $\varphi : X \to R$ and $r > 0$ we define a subset $S_{\varphi,r} \subset X$ as follows:

$$S_{\varphi,r} = \{x \in X \mid\mid \varphi(x) \mid > r\}.$$

Also, we will observe the following vector space of real-valued functions:

$$Z(X,R) = \{\varphi \in^X R \mid S_{\varphi,r} \quad \text{is finite for all} \quad r > 0\}.$$

Theorem 2.2. *Let $(X, \mathcal{O})$ be a compact Hausdorff space. Then*

$$\Lambda(X,R) = C(X,R) \oplus Z(X,R).$$

Proof. Let $\varphi \in Z(X,R), x_0 \in X$ and $\epsilon > 0$. Then $W = (X \setminus S_{\varphi,\epsilon/2}) \cup \{x_0\}$ is a cofinite set containing x_0, hence $W \in \mathcal{U}(x_0)$. Since $\mid \varphi(x) \mid < \epsilon$ for all $x \in W \setminus \{x_0\}$, we have $\lim_{x_0} \varphi = 0$. So, we proved $Z(X,R) \subset \Lambda(X,R)$. Moreover, $Z(X,R)$ is a subspace of $\Lambda(X,R)$ because for $\varphi, \psi \in Z(X,R)$ and $\alpha \in R$ there holds:

$$S_{\varphi+\psi,r} \subset S_{\varphi,r/2} \cup S_{\psi,r/2} \quad \text{and} \quad S_{\alpha\varphi,r} \subset S_{\varphi,r/|\alpha|}.$$

Suppose $\varphi \in C(X,R) \cap Z(X,R) \setminus \{0\}$. Then, there is $x \in X$ such that $\varphi(x) > 0$ (or $\varphi(x) > 0$) and $\varphi^{-1}((\varphi(x)/2, \infty)) = S_{\varphi,\varphi(x)/2}$ is an open, finite set. Since X has no isolated points this is impossible. Thus, $C(X,R) \cap Z(X,R) = \{0\}$.

Let $f \in \Lambda(X,R)$ and let $F \in C(X,R)$ be the function defined in Corollary 1.1. By Theorem 1.2 (f) the set

$$\Delta_r = \{x \in X \mid\mid F(x) - f(x) \mid > r\} = S_{f-F,r}$$

is finite for each $r > 0$. Thus, $\varphi = f - F \in Z(X,R)$ and $f = F + \varphi$. Each function $f \in \Lambda(X,R)$ has the (unique) representation $f = F + \varphi$, where $F \in C(X,R)$ and $\varphi \in Z(X,R)$. □

3. Bounded linear functionals on $\Lambda(I, R)$

By I we will denote an arbitrary segment $[a, b] \subset R$. The proof of the following lemma is elementary.

Lemma 3.1. *(i)* $\sup(A \cup B) = \max\{\sup A, \sup B\}$, *for all nonempty, bounded* $A, B \subset R$.

(ii) If $F \in C(I, R)$ *and* $P \subset I$ *is a countable set, then* $\sup_{x \in I} \mid F(x) \mid = \sup_{x \in I \setminus P} \mid F(x) \mid$. □

Theorem 3.1. *The functional* $\Psi : \Lambda(I, R) \to R$ *is bounded and linear iff there are the unique bounded, linear functionals* $\Gamma : C(I, R) \to R$ *and* $\Phi : Z(I, R) \to R$ *such that for all* $F \in C(I, R)$ *and all* $\varphi \in Z(I, R)$

$$\Psi(F + \varphi) = \Gamma(F) + \Phi(\varphi).$$

Proof. ($\Rightarrow$) Consider the restrictions $\Gamma = \Psi \mid C(I, R)$ and $\Phi = \Psi \mid Z(I, R)$.

($\Leftarrow$) The linearity of Ψ is obvious. Let $f = F + \varphi \in \Lambda(I, R)$. By the previous lemma we have

$$\|f\| = \sup\{\mid F(x) + \varphi(x) \mid : x \in I\}$$

$$= \sup(\{\mid F(x) + \varphi(x) \mid : x \in \varphi^{-1}(R \setminus \{0\})\} \cup \{\mid F(x) \mid : x \in \varphi^{-1}(0)\})$$

$$= \max\{\sup_{\varphi(x) \neq 0} \mid F(x) + \varphi(x) \mid, \sup_{\varphi(x) = 0} \mid F(x) \mid\}$$

$$\geq \sup_{\varphi(x) = 0} \mid F(x) \mid = \sup_{x \in I} \mid F(x) \mid = \|F\|$$

because $\{x \in I \mid \varphi(x) \neq 0\} = \bigcup_{n \in N} S_{\varphi, 1/n}$ is a countable set. Also, $\|\varphi\| \leq \|\varphi + F\| + \|F\| = \|f\| + \|F\| \leq 2\|f\|$ and Ψ is bounded because

$$\mid \Psi(f) \mid = \mid \Psi(F + \varphi) \mid = \mid \Gamma(F) + \Phi(\varphi) \mid \leq \|\Gamma\| \|F\| + \|\Phi\| \|\varphi\| \leq$$

$$\leq (\|\Gamma\| + 2\|\Phi\|)\|f\|. \quad \square$$

It is well-known that the set of real-valued functions $\ell_1(I, R)$ defined by:

$$\ell_1(I, R) = \{h \in^I R : \mid \{x \in I : h(x) \neq 0\} \mid \leq \omega \wedge \sum_{x \in I} \mid h(x) \mid < \infty\}$$

is a vector space with the norm

$$\|h\| = \sum_{x\in I} | h(x) |, \quad h \in \ell_1(I,R).$$

Also, the following statement is folklore.

Theorem 3.2. *Let $h \in \ell_1(I,R)$. Then the functional $\Phi : Z(I,R) \to R$ given by*

$$(**) \qquad \Phi(\varphi) = \sum_{x\in I} \varphi(x)h(x), \quad \varphi \in Z(I,R)$$

is a bounded linear functional on $Z(I,R)$.

Converselly, if $\Phi : Z(I,R) \to R$ is a bounded, linear functional, then there is the unique $h \in \ell_1(I,R)$ satisfying $\|h\| = \|\Phi\|$ and (**) . □

If $BV(I,R)$ is the space of all real-valued functions of bounded variation, whose domain is $I = [a,b]$ and

$$NV_0(I,R) = \{v \in BV(I,R) : v(a) = 0 \wedge \forall x \in I \setminus \{a\} \;\; v(x-0) = v(x)\}$$

then we have the following consequence of Theorems 3.1, 3.2 and the well-known Riesz Theorem:

Theorem 3.3. *Let $\Psi : \Lambda(I,R) \to R$ be a bounded, linear functional. Then, there are the unique $v \in NV_0(I,R)$ and $h \in \ell_1(I,R)$ such that for each $f = F + \varphi \in \Lambda(I,R)$*

$$(11) \qquad \Psi(F+\varphi) = \int_I F dv + \sum_{x\in I} \varphi(x)h(x).$$

Conversely, for each $v \in NV_0(I,R)$ and $h \in \ell_1(I,R)$ the mapping Ψ, defined by (11), is a bounded, linear functional on $\Lambda(I,R)$. ($\int_I F dv$ is the Riemann-Stieltjes integral). □

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