NOVI SAD J. MATH.
VOL. 27, NO. 1, 1997, 1-17

DIFFERENT ACCELERATION PROCEDURES OF NEWTON'S METHOD

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Abstract

Three different procedures to accelerate Newton's method are considered. All of them are based on the influence of convexity of a real function. From these accelerations we define three iterative methods and a family of iterations. Finally, we see which of the methods show the fastest convergence to a solution of a nonlinear equation.

AMS Mathematics Subject Classification (1991): 65H05
Key words and phrases: Nonlinear equations, Newton method, Halley method, Convex Acceleration of Newton's method, Chebyshev method, convexity

1. Introduction

We analyse the influence of convexity of a curve $y = f(x)$ in the velocity of convergence for the sequence defined by Newton's method (second order):

$$x_{n+1} = F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0. \tag{1}$$

To measure the convexity of the curve $y = f(x)$ we use the degree of logarithmic convexity [11]:

$$L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}, \tag{2}$$

if $f'(x) \neq 0$. It is a measure of convexity at each point of the curve. Observe that $F'(x) = L_f(x)$, where F is defined in (1).

Taking into account the geometrical interpretation of Newton's method [15], we deduce that the smaller convexity of the curve $y = f(x)$, the faster is the convergence rate of sequence (1) to a unique solution x^* of equation

$$f(x) = 0. \tag{3}$$

In particular, if $y = f(x)$ is a line, we have $x_1 = x^*$.

Three procedures of acceleration of Newton's method are given. First one consists of reducing directly the degree of logarithmic convexity of f. From the function f, two functions with a lower degree of logarithmic convexity than f are provided. In the second procedure, named global approximation, the curve $y = f(x)$ is approximated by the tangent line at $(x^*, 0)$. Note that lines are the curves with a lower degree of logarithmic convexity. Finally, in the procedure called local approximation the curve $y = f(x)$ is approximated by means of the line $y = f'(x_n)(x_n - x^*)$ in a neighbourhood of each point $(x_n, f(x_n))$.

By these three procedures, pointwise accelerations are obtained. Consequently, independient iterative processes are thus defined.

2. Influence of convexity in Newton's method

We extend the justification of the above-mentoned ideas and complete a study given by Hernández in [12]. Let f be a real function defined on an interval $[a, b]$. Let us assume that f satisfies the following Fourier conditions in $[a, b]$: $f(a) < 0 < f(b), f'(x) > 0$ and $f''(x) \geq 0$. For other situations, it suffices to change $f(x)$ for $f(-x), -f(x)$ or $-f(-x)$. It is obvious that in the above conditions, there exists a unique solution x^* of equation (3). Set $x_0 \in [a, b]$ with $f(x_0) > 0$.

On the other hand, let g be a function satisfaying the same conditions as f in $[a, b]$. Let us assume that x^* is also the unique solution of $g(x) = 0$ in $[a, b]$. Let $y_0 = x_0$, and consider for all $n \geq 0$ the sequence

$$y_{n+1} = G(y_n) = y_n - \frac{g(y_n)}{g'(y_n)}. \tag{4}$$

By the degree of logarithmic convexity of functions f and g, we will compare the velocity of convergence of sequences (1) and (4).

When the velocity of convergence of two sequences with the same order of convergence is compared, the problem of accessibility to the solution x^* of equation (3) appears. So, we study first this problem for Newton's method.

Lemma 2.1. *Let f satisfy the above Fourier conditions in $[a, b]$. Let $x_0 \in [a, b]$ with $f(x_0) > 0$ and $\{x_n\}$ be the sequence defined by (1). If $x_{n-1} \neq x^*$ and $x_n = x^*$, then $f(x) = ax + b$ with $a, b \in \mathbf{R}$, for all $x \in (x^*, x_{n-1})$. Furthermore $x_{n+k} = x^*$ for all $k \in \mathbf{N}$.*

Proof. From F defined in (1) we deduce that $F(x) \leq x^*$ in (x^*, x_{n-1}), since F is a nondecreasing function in $[x^*, b]$. By means of Mean Value theorem, we get

$$F(x) - x^* = F(x) - F(x^*) = F'(w)(x - x^*)$$

for some $w \in (x^*, x)$, and then $F(x) \geq x^*$. Therefore, $F(x) = x^*$ for $x \in (x^*, x_{n-1})$ and consequently, $f(x) = ax + b$ in (x^*, x_{n-1}) with $a, b \in \mathbf{R}$.

On the other hand, it is clear that $x_{n+k} = x^*$ for all $k \in \mathbf{N}$, since x^* is a fixed point of F. □

Now we compare the velocity of sequences defined in (1) and (4) with a result given by Hernández [12].

Theorem 2.1. *Let f and g satisfy the above Fourier conditions in $[a, b]$ and both of them have the same solution x^* in $[a, b]$. Let $f(x_0) > 0$ and $g(x_0) > 0$ for $x_0 \in [a, b]$. If $L_f(x) > L_g(x) > 0$ in $[a, b] - \{x^*\}$, then sequence (4) converges to x^* faster than sequence (1) for $y_0 = x_0$.*

Notice that if we consider $f(x_0) < 0$, we have to assure that $x_1 \leq b$ and $y_1 \leq b$ in order to guarantee that sequences (1) and (4) lie in $[a, b]$. Then both sequences will be decreasing from x_1 and y_1, since that $L_f(x) < L_g(x)$ in $[x_0, x^*)$. Therefore, with slight modifications we obtain an analogous result to theorem 2.1.

Example 1. Let $f(x) = \dfrac{x^3}{216} - 1$ and $g(x) = \dfrac{x^2}{36} - 1$ be two functions with the same zero $x^* = 6$ in $[3,10]$. From (2) we obtain that

$$L_f(x) = \frac{2}{3} - \frac{144}{x^3} \quad \text{and} \quad L_g(x) = \frac{1}{2} - \frac{18}{x^2}.$$

These functions are increasing and convex in $[3,10]$. Therefore, by theorem 1.2, we have that the sequence $\{y_n\}$ converges to $x^* = 6$ faster that $\{x_n\}$ if $y_0 = x_0 \in [3,10]$ and $f(x_0) > 0$.

On the other hand, if we choose $x_0 = 2 \in [3,10]$ where $f(2) < 0$, then $10 \geq F(3) = 10$ and $10 \geq G(3) = 7.5$. Therefore, $x_n, y_n \in [6,10]$ for all $n \geq 1$, see Table 1.

n	x_n	y_n
0	3.0000000000000	3.0000000000000
1	10.0000000000000	7.5000000000000
2	7.3866666666667	6.1500000000000
3	6.2440237430147	6.0018292682927
4	6.0094124974239	6.0000002787669
5	6.0000147350265	6.0000000000000
6	6.0000000000362	6.0000000000000

Table 1

Next, we extend the previous situation. The following result solves the problem that Fourier conditions are insufficient to get convergence for Newton's method when f'' changes the sign in $[a,b]$.

Theorem 2.2. *Let f be a derivable enough function that satisfies $f(a)f(b) < 0$, $f'(x) \neq 0$ and $|L_f(x)| \leq M < 1$ in $[a,b]$. Then, the iterative method defined by (1) is convergent to x^* for any $x_0 \in [a,b]$, where $a \leq F(x_0) \leq b$.*

Proof. Since F is a contractive function, it follows that $|x_2 - x^*| < |x_1 - x^*|$. By mathematical induction, it is easy to check that $|x_n - x^*| \leq M^{n-1}|x_1 - x^*| < |x_1 - x^*|$ for all $n \in \mathbb{N}$. Besides, from $x_1 \in [a,b]$ it follows that $x_n \in [a,b]$ for all $n \in \mathbb{N}$. Now, the convergence of (1) to x^* is deduced in a similar way. □

Note that the last result allows improvement of the widely used Fourier conditions.

Example 2. Let us consider $f(x) = -x^3 + 3x^2 - 2$ in the interval $\left[\frac{1}{10}, \frac{19}{10}\right]$. Note that this function has an inflexion point in $\left[\frac{1}{10}, \frac{19}{10}\right]$. Hence f'' changes the sign in that interval. The function defined from (2) is $L_f(x) = \frac{6(-x^3+3x^2-2)(1-x)}{-3x^2+6x}$ and it satisfies $|L_f(x)| < 1$ in $\left[\frac{1}{10}, \frac{19}{10}\right]$. If we choose $x_0 = 1.6$, then $x_1 = 0.775 \in \left[\frac{1}{10}, \frac{19}{10}\right]$. Therefore, by Theorem 2.3, the sequence $\{x_n\}$ given by (1), converges to the solution $x^* = 1$ in $\left[\frac{1}{10}, \frac{19}{10}\right]$, see Table 2.

n	x_n
0	1.6000000000000000
1	0.7750000000000000
2	1.0079986833443050
3	0.9999996588133421
4	1.0000000000000000

Table 2

Example 3. Consider $f(x) = \frac{1}{2} + \sin x$. Then $L_f(x) = -\frac{\sin x(1+2\sin x)}{2\cos^2 x}$. It is easy to check that $|L_f(x)| < 1$ in $[-1.00297, 0.634867]$. We choose $x_0 = 0.6$ for theorem 2.3 and obtain that $x_1 = -0.6899 \in [-1.00297, 0.634867]$. Consequently, the sequence $\{x_n\}$ defined by (1), lies in $[-1.00297, 0.634867]$, and converges to the solution $x^* = -0.5235987755982988711$, see Table 3.

n	x_n
0	0.6000000000000000000
1	-0.6899509655978506667
2	-0.5129726247150719697
3	-0.5235667752006047706
4	-0.5235667753027045709
5	-0.5235667755982988737
6	-0.5235667755982988705
7	-0.5235667755982988742
8	-0.5235667755982988711
9	-0.5235667755982988711

Table 3

Note that Fourier conditions are satisfied in $[-\frac{\pi}{2}, \varepsilon]$, where $\varepsilon < 0$. Therefore, we have extended the domain of inital points to apply Newton's method.

After proving that convexity of the function f influences the rate of convergence of Newton's method, we get different sequences that converge faster than Newton's one. We call these sequences accelerations of Newton's iteration and they are pointwise accelerations. That is to say, we obtain $y_{n+1} = G(x_n)$ where G is defined in (4) so that y_n is closer to x^* than x_n. It is a consequence of the fact that we will be able to define new iterative processes of the form $z_{n+1} = G(z_n)$ from the function G.

3. Convex acceleration of Newton's method by means of direct reduction of the degree of logarithmic convexity of the function

Throughout this section f and g are two increasing and convex functions in $[a, b]$ that satisfy $|L_f(x)| > |L_g(x)|$ for $x \in (x^*, x_0]$. Let $x_0 \in [a, b]$ with $f(x_0) > 0$. We analyse two different ways to accelerate Newton's sequence.

Firstly, we consider the function [13]

$$g(x) = \frac{f(x)}{1 + \alpha f(x)}$$

where $1 + \alpha f(x) > 0$ in $[a, b]$ and $\alpha \geq 0$. It is obvious that $g(x) > 0$ in $(x^*, b]$ under the previous statements. Then we have

$$g'(x) = \frac{f'(x)}{(1 + \alpha f(x))^2}$$

and

$$g''(x) = \frac{f''(x)(1 + \alpha f(x)) - 2\alpha f'(x)^2}{(1 + \alpha f(x))^3}.$$

As $L_f(x^*) = 0$, there exists an $x_0 \in [a, b]$, where $L_f(x) < 2$ in $(x^*, x_0]$. Moreover, if α satisfies

$$\alpha \leq \min_{x \in (x^*, x_0]} \left\{ \frac{U[f](x)}{2 - L_f(x)} \right\}$$

where $U[f](x) = \frac{f''(x)}{f'(x)^2}$, then g is increasing and convex in $(x^*, x_0]$ (this function and its connection with Whittaker's method is studied in [10]). Otherwise

$$L_g(x) = L_f(x) - \alpha f(x)(2 - L_f(x))$$

and $L_f(x) > L_g(x) > 0$ in $(x^*, x_0]$. So, we define a uniparametric family of accelerations of sequence (1):

$$y_{\alpha,n+1} = x_{\alpha,n} - \frac{g(x_{\alpha,n})}{g'(x_{\alpha,n})} = x_{\alpha,n} - \frac{f(x_{\alpha,n})}{f'(x_{\alpha,n})}(1 + \alpha f(x_{\alpha,n})), \quad n \geq 0. \tag{5}$$

From Theorem 2.2 it follows that (5) converges to x^* faster than Newton's sequence $\{x_n\}$. Moreover (5) is a family of accelerations, since (1) and (5) are decreasing, and $y_{\alpha,n} < x_{\alpha,n}$ for all $\alpha \geq 0$ and $n \in \mathbf{N}$.

Secondly, we consider the function

$$g(x) = \frac{f(x)}{\sqrt{f'(x)}}$$

introduced by Alefeld [1] to define Halley's method as a variant of Newton's method. Thus

$$g'(x) = \sqrt{f'(x)}\left(1 - \frac{1}{2}L_f(x)\right)$$

and

$$g''(x) = \frac{f''(x)L_f(x)}{2\sqrt{f'(x)}}\left(\frac{3}{2} - L_f(x)\right).$$

So if $L_f(x) < 2$ and $L_{f'}(x) \leq \frac{3}{2}$ in $(x^*, x_0]$, then g is increasing and convex in $(x^*, x_0]$. As

$$L_g(x) = \frac{L_f(x)^2}{(L_f(x) - 2)^2}(3 - 2L_{f'}(x)),$$

hence $L_f(x) > L_g(x) > 0$ in $(x^*, x_0]$ if and only if $(L_f(x) - 2)^2 - L_f(x)(3 - 2L_{f'}(x)) > 0$. To get $L_f(x) > L_g(x)$ in $(x^*, x_0]$ it is enough to choose x_0 so that

$$L_f(x) < \frac{1}{2}\left(7 - 2m - \sqrt{(7 - 2m)^2 - 16}\right)$$

in $(x^*, x_0]$ where $m = \min_{x \in [x^*, b]} \{L_{f'}(x)\}$. Under these assumptions, we obtain by Theorem 2.2 an acceleration of Newton's method defined for all $n \geq 0$ by

$$y_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{2}{2 - L_f(x_n)}\right). \tag{6}$$

Example 4. Let us consider the function $f(x) = e^x + x$. We see from Table 4 that sequence $\{y_n\}$ defined by (6) converges to the solution $x^* = -0.5671432904097839$ of equation $f(x) = 0$ in $[-2, 2]$. Besides, it is an acceleration of the sequence (1) starting from $x_0 = 2$. We obtain that

$$L_f(x) = \frac{e^x(e^x + x)}{(e^x + 1)^2}, \quad L_{f'}(x) = \frac{1 + e^x}{e^x}, \quad m = L_f(2) = 1.13534$$

and

$$\frac{1}{2}\left(7 - 2m - \sqrt{(7 - 2m)^2 - 16}\right) = 1.10306.$$

Thus, it is easy to check that $L_f(x) < 1.10306$ in **R**.

n	x_n	y_n
0	2.0000000000000000	2.0000000000000000
1	0.8807970779778824	-0.2070451959228786
2	-0.0842749600983386	-0.5683407447276397
3	-0.5193066837383489	-0.5671432903624338
4	-0.5667232231976213	-0.5671432904097839
5	-0.5671432584762297	-0.5671432904097839
6	-0.5671432904097837	-0.5671432904097839

Table 4

4. Global convex acceleration of Newton's method

This acceleration procedure was introduced by Hernández in [12]. The curve $y = f(x)$ is approximated by the tangent line at $(x^*, 0)$, (see Fig. 1). For that, it suffices to consider $f \in C^2[a, b]$ and the limited Taylor's formula of f in a neighbourhood of x^*:

$$f(x) \sim f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2.$$

Then, $f'(x^*)(x - x^*) \sim f(x) - \dfrac{f''(x^*)}{2!}(x - x^*)^2$, and we consider the function

$$g(x) = f(x) - \frac{f''(x^*)}{2!}(x - x^*)^2.$$

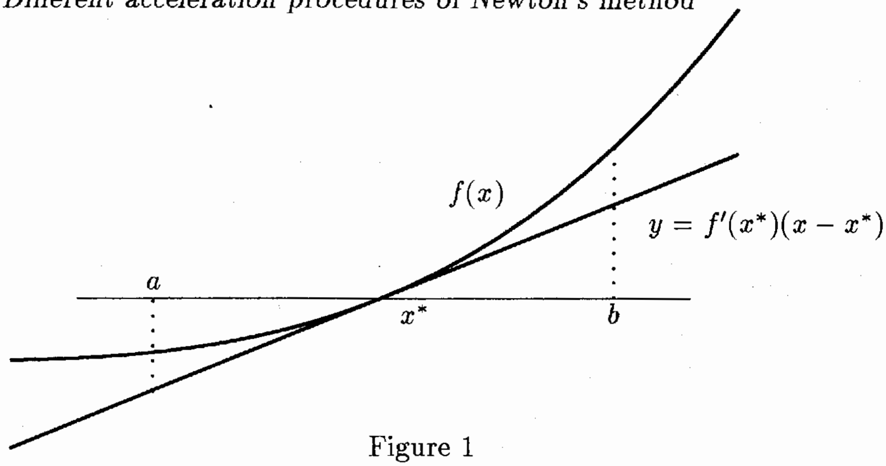

Figure 1

But this function runs into a problem: x^* is an unknown value. As our purpose is to accelerate the sequence (1), we must obtain a new value $y_{n+1} = G(x_n)$, where G is defined in (4), closer to x^* than x_{n+1}. Thereby, we have to evaluate g and g' at x_n. So we approximate

(i) $f''(x^*) \sim f''(x_n)$,

(ii) $(x_n - x^*)^k \sim (x_n - x_{n+1})^k$ for $k = 1, 2$, since $\lim_n \frac{(x_n - x^*)^k}{(x_n - x_{n+1})^k} = 1$.

Therefore, we consider

$$g(x_n) \sim f(x_n) - \frac{f''(x_n)}{2!}\left(\frac{f(x_n)}{f'(x_n)}\right)^2$$

and

$$g'(x_n) \sim f'(x_n) - f''(x_n)\frac{f(x_n)}{f'(x_n)}$$

in order to define the sequence

$$y_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \sim x_n - \frac{f(x_n)}{2f'(x_n)}\left(1 + \frac{1}{1 - L_f(x_n)}\right). \qquad (7)$$

To see that (8) is an acceleration of (1) [12], it is suffices to show that

$$\lim_n \frac{|y_n - x^*|}{|x_n - x^*|} = 0.$$

Example 5. Given the function $f(x) = \dfrac{e^x - 5x}{x}$, we show in Table 5 that sequence (7) converges to the solution $x^* = 2.542641357773526$ of $f(x) = 0$ in $[1, 4]$ faster than Newton's one.

n	x_n	y_n
0	3.500000000000000	3.500000000000000
1	2.839835893846803	2.441271065123373
2	2.577023717097117	2.542750966419476
3	2.543144242829421	2.542641357773588
4	2.542641466706540	2.542641357773526
5	2.542641357773532	2.542641357773526

Table 5

5. Local convex acceleration of Newton's method

The third acceleration procedure, that we call local approximation, consists of approximating the curve $y = f(x)$ by lines. When the sequence $\{x_n\}$ given by (1) is obtained, for each point x_n of (1) the curve $y = f(x)$ is approximated by the line $y = f'(x_n)(x - x^*)$ in a neighbourhood of x_n, see fig. 2.

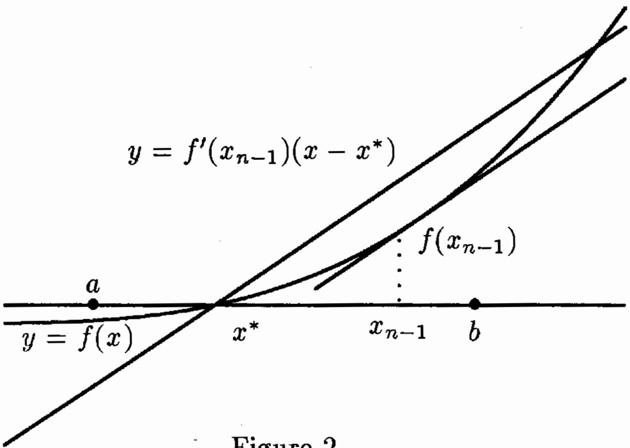

Figure 2

Let us consider $f \in C^2[a, b]$ and the limited Taylor's formula

$$f(x) \sim f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2!}(x - x_n)^2. \tag{8}$$

So we consider $g(x) = f'(x_n)(x - x^*)$. Now we have to approximate $g(x_n)$ and $g'(x_n)$, to get $y_{n+1} = G(x_n)$, where G is given in (4).

Taking into account (8), for $x = x^*$:

$$g(x_n) \sim f(x_n) + \frac{f''(x_n)}{2!}(x^* - x_n)^2$$

and

$$(x^* - x_n)^2 \sim (x_{n+1} - x_n)^2 = \left(\frac{f(x_n)}{f'(x_n)}\right)^2,$$

we obtain

$$g(x_n) \sim f(x_n)\left(1 + \frac{1}{2}L_f(x_n)\right).$$

On the other hand, it is obvious that $g'(x_n) = f'(x_n)$. Therefore

$$y_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \sim x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{1}{2}L_f(x_n)\right). \tag{9}$$

Using the same argument as for sequence (7), it is proved that (9) is an acceleration of (1).

Example 6. Let us consider again $f(x) = \dfrac{e^x - 5x}{x}$. We see from Table 6 that the sequence $\{y_n\}$ defined by (9) converges to $x^* = 2.542641357773526$ faster than (1).

n	x_n	y_n
0	3.500000000000000	3.500000000000000
1	2.839835893846803	2.659283282924826
2	2.577023717097117	2.543020336792808
3	2.543144242829421	2.542641357787998
4	2.542641466706540	2.542641357773526
5	2.542641357773532	2.542641357773526

Table 6

6. Iterative processes obtained by means of convex acceleration of Newton's method

We have obtained four accelerations of Newton's method as a consequence of convexity of the function f. All of them have the form $y_{n+1} = G(x_n)$, where G is defined in (4) and, therefore, independent iterative processes may be considered. In this section we define iterative processes, and give a first study of the convergence according to the degree of logarithmic convexity of f.

The acceleration (5) provides the family of iterations

$$x_{\alpha,n+1} = x_{\alpha,n} - \frac{f(x_{\alpha,n})}{f'(x_{\alpha,n})}\left(1 + \alpha f(x_{\alpha,n})\right), \quad n \geq 0. \tag{10}$$

Let us consider the f increasing and convex in $[a,b]$. Denote

$$< a, b, x_{\alpha,0} >= \begin{cases} [x_{\alpha,0}, b] & if\ x_{\alpha,0} < x^* \\ [a, x_{\alpha,0}] & if\ x_{\alpha,0} > x^* \end{cases}$$

and

$$m < a, b, x_{\alpha,0} >= \min_{x \in <a,b,x_{\alpha,0}>} \left\{ \frac{U[f](x)}{2 - L_f(x)} \right\}.$$

Then, we have the following result that has been proved in [13].

Theorem 6.1. *Let $x_{\alpha,0} \in [a,b]$ with $f(x_{\alpha,0}) > 0$ and $|L_f(x)| < 2$ in $< a, b, x_{\alpha,0} >$. If $0 \leq \alpha \leq m < a, b, x_{\alpha,0} >$, then the sequence given by (10) is decreasing and converges quadratically to x^*. Moreover, if $0 \leq \alpha < \beta \leq m < a, b, x_{\alpha,0} >$, the sequences $\{x_{\beta,n}\}$ converges to x^* faster than $\{x_{\alpha,n}\}$.*

If $f(x_{\alpha,0}) < 0$ and $b - x_{\alpha,0} \geq -\frac{f(x_{\alpha,0})}{f'(x_{\alpha,0})}$, under the same assumptions as mentioned above in Theorem 6.1, we get an analogous result.

An interesting feature of the family (10) is that for a similar efficiency index to Newton's method, we obtain an iterative process that converges to x^* faster than Newton's iteration.

Example 7. Let us consider the function $f(x) = \ln\left(\dfrac{2}{2-x}\right)$ in $\left[-\frac{3}{2}, \frac{3}{2}\right]$. Then, $U[f](x) = 1$ and $L_f(x) = \ln\left(\frac{2}{2-x}\right)$. Observe that the function $\frac{U[f](x)}{2-L_f(x)}$

is nondecreasing in **R**. From Theorem 6.1 it follows that the fastest iterative method of (10) is given by $\alpha = \frac{U[f](-\frac{3}{2})}{2-L_f(-\frac{3}{2})} = 0.390648$. Therefore, the iterative process of the family (10) where $\alpha = 0.390648$ is given by the sequence $\{x_{0.390648,n}\}$. From Table 7, we see that the previous sequence converges to the solution $x^* = 0$ of equation $f(x) = 0$ faster than Newton's sequence $\{x_{0,n}\}$.

n	$x_{0,n}$	$x_{0.390648,n}$
0	1.500000000000000000	1.500000000000000000
1	0.806852819440054700	0.431442208860817500
2	0.190529451739077100	0.014114389234717540
3	0.009378120633087785	0.000011006483878148
4	0.000022021734024151	0.000000000000000000

Table 7

The acceleration (6) provides the well known Halley's method or method of tangent hyperbolas ([1],[2],[4],[6],[14],[16]):

$$x_{n+1} = F_1(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \frac{2}{2 - L_f(x_n)}, \quad n \geq 0. \tag{11}$$

The global approximation procedure provides the iterate

$$x_{n+1} = F_2(x_n) = x_n - \frac{f(x_n)}{2f'(x_n)} \left(1 + \frac{1}{1 - L_f(x_n)}\right), \quad n \geq 0, \tag{12}$$

named Convex Acceleration of Newton's method or Super-Halley method ([7],[9],[12]).

Finally, by local approximation procedure we obtain the Chebyshev iteration ([3],[5]):

$$x_{n+1} = F_3(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{1}{2} L_f(x_n)\right), \quad n \geq 0. \tag{13}$$

On the other hand, it is well-known [8] that a method given by the expression

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H(L_f(x_n)), \quad n \geq 0,$$

where $H(0) = 1$, $H'(0) = \frac{1}{2}$ and $|H''(0)| < \infty$ is of third order. Therefore (11), (12) and (13) are cubically convergent.

Now we give a convergence result depending on L_f and $L_{f'}$ for the previous three methods. Consider the function f is increasing, convex, and derivable enough in $[a, b]$

Theorem 6.2. *Let $x_0 \in [a, b]$ with $f(x_0) > 0$ and $L_{f'}(x) \leq 0$ in $[a, b]$.*

(i) *If $L_f(x) < 2$ in $[a, b]$, then the sequence $\{x_n\}$ given by (11) is decreasing and converges to x^*.*

(ii) *If $L_f(x) < 1$ in $[a, b]$, then the sequence $\{x_n\}$ given by (12) is decreasing and converges to x^*.*

(iii) *The sequence $\{x_n\}$ given by (13) is decreasing and converges to x^*.*

Proof. It follows $x_0 \geq x^*$ from $f(x_0) > 0$. By Mean Value theorem we have

$$x_1 - x^* = F_1'(w_0)(x_0 - x^*)$$

for some $w_0 \in (x^*, x_0)$. Taking into account that

$$F_1'(x) = \left(\frac{L_f(x)}{2 - L_f(x)}\right)^2 (3 - 2L_{f'}(x)),$$

we deduce that $F_1'(x) \geq 0$ in $[x^*, b]$. So $x_1 \geq x^*$. Now it is easy to prove by mathematical induction that $x_n \geq x^*$ for all $n \in \mathbf{N}$.

Moreover

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} \frac{2}{2 - L_f(x_n)} \leq 0$$

for all $n \geq 0$.

Thereby, the sequence (11) is decreasing and converges to $u \in [a, b]$, where $u \geq x^*$. Making $n \to \infty$ in (11), we get

$$u = u - \frac{f(u)}{f'(u)} \frac{2}{2 - L_f(u)}$$

and consequently $f(u) = 0$. But under the above hypotheses, x^* is the unique solution of equation (3) in $[a, b]$, therefore $u = x^*$ and **(i)** is proved.

On the other hand, taking into account that

$$F_2'(x) = \frac{L_f(x)^2}{2(1-L_f(x))^2}(L_f(x) - L_{f'}(x))$$

and

$$F_3'(x) = \frac{1}{2}L_f(x)^2(3 - L_{f'}(x)),$$

the cases **(ii)** and **(iii)** are analogous to the previous one (see [12] for **(ii)**).
□

Note that the obtained sequences defined by (11), (12) and (13) are increasing and converge to x^* under the same assumptions of the previous theorem but for $f(x_0) < 0$.

Furthermore, we deduce the same result by the Chebyshev method if $L_{f'}(x) \leq 3$ for $x \in [a,b]$, and Halley method if $L_f(x) < 2$ and $L_{f'}(x) \leq \frac{3}{2}$ in $[a,b]$. However, to compare the rates of convergence of the three methods we need the same assumptions to get simultaneous convergence for (11), (12), and (13).

Theorem 6.3. *Under assumtions of Theorem 6.2 and starting from the same initial point, the rate of convergence of sequence (12) is higher than the one of sequence (11), and the latter one is higher than for sequence (13).*

Proof. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by (12), (13) and (14) respectively. Since $x_0 = y_0 = z_0$ and the three sequences are decreasing, we expect that $z_n \leq x_n \leq y_n$ for all $n \in \mathrm{N}$. This can be proved by mathematical induction. Let $n = 1$, then

$$y_1 - x_1 = F_2(x_0) - F_1(x_0) = \frac{f(x_0)}{f'(x_0)}\left(\frac{2}{2-L_f(x_0)} - \frac{2-L_f(x_0)}{2(1-L_f(x_0))}\right) \leq 0$$

and

$$x_1 - z_1 = F_1(x_0) - F_3(x_0) = \frac{f(x_0)}{f'(x_0)}\left(1 + \frac{1}{2}L_f(x_0) - \frac{2}{2-L_f(x_0)}\right) \leq 0.$$

Now we assume that $z_{n-1} \leq x_{n-1} \leq y_{n-1}$. Taking into account that F_1 and F_3 are non-increasing functions in $[a,b]$, it is easy to show that

$$y_n - x_n = F_2(y_{n-1}) - F_1(x_{n-1}) \leq F_2(y_{n-1}) - F_1(y_{n-1}) \leq 0$$

and

$$x_n - z_n = F_1(x_{n-1}) - F_3(z_{n-1}) \leq F_1(x_{n-1}) - F_3(x_{n-1}) \leq 0$$

So, the induction is completed. □

Example 8. Consider the function $f(x) = x - \cos x$. This function is increasing and convex in $[0, \frac{\pi}{2}]$. From (2) it follows that

$$L_f(x) = \frac{(x - \cos x)\cos x}{(1 + \sin x)^2} \quad \text{and} \quad L_{f'}(x) = \frac{\sin x}{\sin x - 1},$$

then $L_f(x) < 1$ and $L_{f'}(x) \leq 0$ in $[0, \frac{\pi}{2}]$. By Theorem 6.2 the sequences (11), (12) and (13) converge to the solution $x^* = 0.7390851332151606428$ of $f(x) = 0$ in $[0, \frac{\pi}{2}]$. In Table 8 we compare convergence rates of the three sequences.

n	z_n	x_n	y_n
0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
1	0.7412215390677832763	0.7408739950803435706	0.7404989832636941698
2	0.7390851348155419594	0.7390851338775818840	0.7390851334050131377
3	0.7390851332151606451	0.7390851332151606499	0.7390851332151606428
4	0.7390851332151606435	0.7390851332151606428	0.7390851332151606428
5	0.7390851332151606428	0.7390851332151606428	0.7390851332151606428

Table 8

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Received by the editors January 10, 1996.