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FIXED POINTS ON THREE COMPLETE METRIC SPACES

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Abstract

We obtained some related fixed point theorems for three metric spaces.

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The following fixed point theorem was proved by Nung [1].

Theorem 1. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces and suppose T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z, into X satisfying the inequalities*

$$\begin{aligned}
d(RSTx, RSy) &\leq c\max\{d(x, RSy), d(x, RSTx), \rho(y, Tx), \sigma(Sy, STx)\},\\
\rho(TRSy, TRz) &\leq c\max\{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy)\},\\
\sigma(STRz, STx) &\leq c\max\{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\}
\end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \leq c < 1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

The next theorem was proved in [2].

Theorem 2. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} d(RSTx, RSTx') &\leq c\max\{d(x,x'), d(x,RSTx), d(x',RSTx'), \\ &\qquad \rho(Tx,Tx'), \sigma(STx,STx')\}, \\ \rho(TRSy, TRSy') &\leq c\max\{\rho(y,y'), \rho(y,TRSy), \rho(y',TRSy'), \\ &\qquad \sigma(Sy,Sy'), d(RSy,RSy')\}, \\ \sigma(STRz, STRz') &\leq c\max\{\sigma(z,z'), \sigma(z,STRz), \sigma(z',STRz'), \\ &\qquad d(Rz,Rz'), \rho(TRz,TRz')\}, \end{aligned}$$

for all x,x' in X, y,y' in Y and z,z' in Z where $0 \leq c < 1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

We now prove the following related fixed point theorems:

Theorem 3 *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z, and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} d^2(RSy, RSTx) &\leq c\max\{d(x,RSy)\rho(y,Tx), \rho(y,Tx)d(x,RSTx), \\ (1) \qquad &\qquad d(x,RSTx)\sigma(Sy,STx), \sigma(Sy,STx)d(x,RSy)\} \\ \rho^2(TRz, TRSy) &\leq c\max\{\rho(y,TRz)\sigma(z,Sy), \sigma(z,Sy)\rho(y,TRSy), \\ (2) \qquad &\qquad \rho(y,TRSy)d(Rz,RSy), d(Rz,RSy)\rho(y,TRz)\} \\ \sigma^2(STx, STRz) &\leq c\max\{\sigma(z,STx)d(x,Rz), d(x,Rz)\sigma(z,STRz), \\ (3) \qquad &\qquad \sigma(z,STRz)\rho(Tx,TRz), \rho(Tx,TRz)\sigma(z,STx)\} \end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \leq c < 1$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively by

$$x_n = (RST)^n x_0, \quad y_n = Tx_{n-1}, \quad z_n = Sy_n$$

for $n = 1, 2, \ldots$.

Applying inequality (2) we have

$$\begin{aligned}\rho^2(y_n, y_{n+1}) &= \rho^2(TRz_{n-1}, TRSy_n)\\ &\leq c\max\{\rho(y_n, y_n)\sigma(z_{n-1}, z_n), \sigma(z_{n-1}, z_n)\rho(y_n, y_{n+1}),\\ &\qquad \rho(y_n, y_{n+1})d(x_{n-1}, x_n), d(x_{n-1}, x_n)\rho(y_n, y_n)\}\\ &= c\max\{\sigma(z_{n-1}, z_n)\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1})d((x_{n-1}, x_n)\}\end{aligned}$$

and so

(4) $$\rho(y_n, y_{n+1}) \leq c\max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}.$$

Applying inequality (3) we have

$$\begin{aligned}\sigma^2(z_n, z_{n+1}) &= \sigma^2(STx_{n-1}, STRz_n)\\ &\leq c\max\{\sigma(z_n, z_n)d(x_{n-1}, x_n), d(x_{n-1}, x_n)\sigma(z_n, z_{n+1}),\\ &\qquad \sigma(z_n, z_{n+1})\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1})\sigma(z_n, z_n)\}\\ &= c\max\{d(x_{n-1}, x_n)\sigma(z_n, z_{n+1}), \sigma(z_n, z_{n+1})\rho(y_n, y_{n+1})\}\end{aligned}$$

and so

$$\begin{aligned}\sigma(z_n, z_{n+1}) &\leq c\max\{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\}\\ &\leq c\max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}\end{aligned} \tag{5}$$

on using inequality (4).

Applying inequality (1) we have

$$\begin{aligned}d^2(x_n, x_{n+1}) &= d^2(RSy_n, RSTx_n)\\ &\leq c\max\{d(x_n, x_n)\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1})d(x_n, x_{n+1}),\\ &\qquad d(x_n, x_{n+1})\sigma(z_n, z_{n+1}), \sigma(z_n, z_{n+1})d(x_n, x_n)\}\\ &= c\max\{\rho(y_n, y_{n+1})d(x_n, x_{n+1}), d(x_n, x_{n+1})\sigma(z_n, z_{n+1})\}\end{aligned}$$

and so

$$\begin{aligned}d(x_n, x_{n+1}) &\leq c\max\{\rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})\}\\ &\leq c\max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}\end{aligned} \tag{6}$$

on using inequalities (4) and (5).

It now follows easily by induction on using inequalities (4), (5) and (6) that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\ \rho(y_n, y_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\ \sigma(z_n, z_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}. \end{aligned}$$

Since $0 \leq c < 1$, it follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits u, v and w in X, Y and Z respectively.

Now suppose that S is continuous. Then $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$ and so

$$Sv = w. \tag{7}$$

Applying inequality (1) we now have

$$\begin{aligned} d^2(RSv, x_{n+1}) &= d^2(RSv, RSTx_n) \\ &\leq c \max\{d(x_n, RSv)\rho(v, Tx_n), \rho(v, Tx_n)d(x_n, x_{n+1}), \\ &\qquad d(x_n, x_{n+1})\sigma(Sv, STx_n), \sigma(Sv, STx_n)d(x_n, RSv)\}. \end{aligned}$$

Letting n tend to infinity, it follows on using equation (7) that $d^2(RSv, u) \leq 0$ and so

$$RSv = u. \tag{8}$$

Using equation (7), this gives us

$$Rw = u. \tag{9}$$

Using equation (8) and inequality (2) we have

$$\begin{aligned} \rho^2(Tu, y_{n+1}) &= \rho^2(TRSv, TRSy_n) \\ &\leq c \max\{\rho(y_n, TRSv)\sigma(Sv, Sy_n), \sigma(Sv, Sy_n)\rho(y_n, TRSy_n), \\ &\qquad \rho(y_n, TRSy_n)d(RSv, RSy_n), d(RSv, RSy_n)\rho(y_n, TRSv)\}. \end{aligned}$$

Letting n tend to infinity, it follows on using equation (8) again that $\rho^2(Tu, v) \leq 0$ and so

$$Tu = v. \tag{10}$$

It now follows from equations (7), (9) and (10) that

$$\begin{aligned} TRSv = TRw = Tu = v, \\ STRw = STu = Sv = w, \\ RSTu = RSv = Rw = u. \end{aligned}$$

The same results of course will hold if R or T is continuous instead of S.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'. Then using inequality (1), we have

$$\begin{aligned} d^2(u,u') &= d^2(RSTu, RSTu') \\ &\le c\max\{d(u',u)\rho(Tu,Tu'), \rho(Tu,Tu')d(u',u'), \\ &\qquad d(u',u')\sigma(STu,STu'), \sigma(STu,STu')d(u',RSTu)\} \\ &= c\max\{d(u,u')\rho(Tu,Tu'), \sigma(STu,STu')d(u,u')\}, \end{aligned}$$

which implies that

$$d(u,u') \le c\max\{\rho(Tu,Tu'), \sigma(STu,STu')\}. \tag{11}$$

Further, using inequality (2), we have

$$\begin{aligned} \rho^2(Tu,Tu') &= \rho^2(TRSTu, TRSTu') \\ &\le c\max\{\rho(Tu',Tu)\sigma(STu,STu'), \sigma(STu,STu')\rho(Tu',Tu'), \\ &\qquad \rho(Tu',Tu')d(u,u'), d(u,u')\rho(Tu,Tu')\} \\ &= c\max\{\rho(Tu,Tu')\sigma(STu,STu'), d(u,u')\rho(Tu,Tu')\}, \end{aligned}$$

which implies that

$$\rho(Tu,Tu') \le c\max\{\sigma(STu,STu'), d(u,u')\}. \tag{12}$$

Inequalities (11) and (12) imply that

$$d(u,u') \le c\sigma(STu,STu'). \tag{13}$$

Finally, using inequality (3), we have

$$\begin{aligned} \sigma^2(STu,STu') &= \sigma^2(STRSTu, STRSTu') \\ &\le c\max\{\sigma(STu,STu')d(u,u'), d(u,u')\sigma(STu',STu'), \\ &\qquad \sigma(STu',STu')\rho(Tu,Tu'), \rho(Tu,Tu')\sigma(STu,STu')\} \\ &= c\max\{\sigma(STu,STu')d(u',u), \sigma(STu,STu')\rho(Tu,Tu')\}, \end{aligned}$$

which implies that

$$\sigma(STu,STu') \le c\max\{d(u,u'), \rho(Tu,Tu')\}. \tag{14}$$

It now follows from inequalities (12), (13) and (14) that

$$d(u,u') \le c\sigma(STu, STu') \le c^2\sigma(STu, STu'),$$

and so $u = u'$, since $c < 1$. The fixed point u of RST is therefore unique. Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of the theorem.

Theorem 4 *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z, and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} & d(RSy, RSTx)\max\{d(x,RSy), d(x,RSTx)\} \\ (15) \qquad & \qquad \le c\sigma(Sy,STx)\max\{\sigma(Sy,STx), d(x,RSTx)\}, \\ & \rho(TRz,TRSy)\max\{\rho(y,TRz),\rho(y,TRSy)\} \\ (16) \qquad & \qquad \le cd(Rz,RSy)\max\{d(Rz,RSy),\rho(y,TRSy)\}, \\ & \sigma(STx,STRz)\max\{\sigma(z,STx),\sigma(z,STRz)\} \\ (17) \qquad & \qquad \le c\rho(Tx,TRz)\max\{\rho(Tx,TRz),\sigma(z,STRz)\} \end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. If one of the mappings R,S,T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

Proof. Let x_0 be an arbitrary point in X and define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively as in the proof of Theorem 2.

Applying inequality (15) we have

$$\begin{aligned} & d(x_n,x_{n+1})\max\{d(x_n,x_n), d(x_n,x_{n+1})\} \\ & \qquad \le c\sigma(z_n,z_{n+1})\max\{\sigma(z_n,z_{n+1}), d(x_n,x_{n+1})\} \end{aligned}$$

and so either

$$d^2(x_n,x_{n+1}) \le c\sigma(z_n,z_{n+1})d(x_n,x_{n+1})$$

which implies that

$$d(x_n,x_{n+1}) \le c\sigma(z_n,z_{n+1})$$

or

$$d^2(x_n,x_{n+1}) \le c\sigma^2(z_n,z_{n+1})$$

which implies that

$$d(x_n, x_{n+1}) \leq b\sigma(z_n, z_{n+1}),$$

where $b = \sqrt{c} \geq c$. Thus either case implies that

$$d(x_n, x_{n+1}) \leq b\sigma(z_n, z_{n+1}). \tag{18}$$

Applying inequality (17) we have

$$\begin{aligned}\sigma(z_n, z_{n+1}) &\max\{\sigma(z_n, z_n), \sigma(z_n, z_{n+1})\} \\ &\leq c\rho(y_n, y_{n+1}) \max\{\rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})\}\end{aligned}$$

and it follows as above that

$$\sigma(z_n, z_{n+1}) \leq b\rho(y_n, y_{n+1}). \tag{19}$$

Applying inequality (16) we have

$$\begin{aligned}\rho(y_n, y_{n+1}) &\max\{\rho(y_n, y_n), \rho(y_n, y_{n+1})\} \\ &\leq cd(x_{n-1}, x_n) \max\{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\}\end{aligned}$$

and it follows as above that

$$\rho(y_n, y_{n+1}) \leq bd(x_{n-1}, x_n). \tag{20}$$

It now follows from inequalities (18), (19) and (20) that

$$d(x_n, x_{n+1}) \leq b\sigma(z_n, z_{n+1}) \leq b^2\rho(y_n, y_{n+1}) \leq \ldots \leq b^{3n} d(x_0, x_1).$$

Since $0 \leq b < 1$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with the limits u, v and w in X, Y and Z respectively.

Now suppose that S is continuous. Then $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$ and so

$$Sv = w. \tag{21}$$

Applying inequality (15) we now have

$$\begin{aligned}d(RSv, x_n) &\max\{d(x_{n-1}, RSv), d(x_{n-1}, x_n)\} \\ &\leq c\sigma(Sv, z_n) \max\{\sigma(Sv, z_n), d(x_{n-1}, x_n)\}.\end{aligned}$$

Letting n tend to infinity, it follows on using equation (21) that $d^2(RSv, u) \leq 0$ and so

$$RSv = u. \tag{22}$$

Using equation (21), this gives us

$$Rw = u. \tag{23}$$

Using equation (23) and inequality (16) we have

$$\begin{aligned} \rho(Tu, y_{n+1}) \max\{\rho(y_n, TRw), \rho(y_n, y_{n+1})\} \\ \leq cd(u, x_n) \max\{d(u, x_n), \rho(y_n, y_{n+1})\}. \end{aligned}$$

Letting n tend to infinity, it follows that $\rho^2(Tu, v) \leq 0$ and so

$$Tu = v. \tag{24}$$

It now follows from equations (21), (23) and (24) that

$$\begin{aligned} TRSv = TRw = Tu = v, \\ STRw = STu = Sv = w, \\ RSTu = RSv = Rw = u. \end{aligned}$$

The same results of course will hold if R or T is continuous instead of S.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'. Then using inequality (15), we have

$$\begin{aligned} d^2(u, u') &= d(RSTu, RSTu') \max\{d(u, RSTu'), d(u, RSTu)\} \\ &\leq c\sigma(STu', STu) \max\{\sigma(STu', STu), d(u, RSTu)\}, \end{aligned}$$

which implies that

$$d(u, u') \leq b\sigma(STu, STu')\}. \tag{25}$$

Further, using inequality (17), we have

$$\begin{aligned} &\sigma(STu, STu') \max\{\sigma(STu, STu'), \sigma(STu, STRSTu)\} \\ &\quad = \sigma(STRSTu, STu') \max\{\sigma(STu, STu'), \sigma(STRSTu, STRSTu)\} \\ &\quad \leq c\rho(Tu', TRSTu) \max\{\rho(Tu', TRSTu), \sigma(STu, STRSTu)\}, \end{aligned}$$

which implies that

$$\sigma(STu, STu') \leq b\rho(Tu, Tu'). \tag{26}$$

Finally, using inequality (16), we have

$$\begin{aligned}\rho(Tu, Tu') \max\{\rho(Tu, TRSTu'), \rho(Tu, TRSTu)\} \\ \leq cd(RSTu', RSTu) \max\{d(RSTu', RSTu), \rho(Tu, TRSTu)\},\end{aligned}$$

which implies that

$$\rho(Tu, Tu') \leq bd(u, u'). \tag{27}$$

Since $b < 1$, it now follows immediately from inequalities (25), (26) and (27) that $u = u'$. The fixed point u of RST is therefore unique. Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of the theorem.

References

[1] Nung, N.P., A fixed point theorem in three metric spaces, Math. Sem. Notes, Kobe Univ. 11 (1983), 77-79.

[2] Jain, R.K., Sahu, H.K, Fisher, B., Related fixed point theorems for three metric spaces, Univ. u Novom Sadu Zb. Prirod.-Mat. Fak. Ser. Mat., to appear.

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