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SEQUENTIAL CONTINUITY AND BOUNDEDNESS FOR THE ADJOINT OPERATORS

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Abstract

Using the Basic matrix Theorem, the sequential continuity of the adjoint operator for linear operator on locally convex topological vector spaces is proved. It is an improvement of the boundedness version of an earlier obtained generalized Adjoint Theorem.

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In [9], we have established a generalization of the theorem (Adjoint Theorem) concerning the boundedness of the adjoint operator for a linear operator on locally convex topological vector space, with respect to particular topologies. Particularly, in the case of the normed space, the adjoint theorem ([1],[7],[8],[11]) asserts that the adjoint operator for a linear operator is always a bounded linear operator when the domain is a $\mathcal{K}$-space ([1], 3.11, [8], [9], [11]).

In this note we prove the sequential continuity of the adjoint operator with respect to particular topologies on locally convex topological vector spaces. Further we improve the boundedness version of the Adjoint Theorem established in [9] by replacing the topology $\mathcal{K}$(E',E) by an other locally convex topology $\beta^*(E', E)$.

Our method of proof is based on the elementary Basic Matrix Theorem ([1], 2.1) so we avoid the use of the Bair Category Theorem (see also [8],[9],[10],[13]).
We begin by fixing some notations and terminology.

If (X, τ) is a topological vector space, a sequence $\{x_k\}$ in X is said to be a τ -$\mathcal{K}$ sequence if every subsequence of $\{x_k\}$ has a further subsequence $\{x_{n_k}\}$ such that the subseries $\sum_k x_{n_k}$ is τ - convergent to an element of X. A topological vector space (X, τ) is said to be a $\mathcal{K}$ - space if every sequence which converges to 0 is a τ - $\mathcal{K}$ sequence ([1], Ch. 3). A subset A of X is said to be τ -$\mathcal{K}$ bounded if for every sequence $\{x_n\} \subseteq A$ and every scalar sequence $\{t_n\}$ which converges to 0, the sequence $\{t_n x_n\}$ is a τ - $\mathcal{K}$ sequence.

Throughout this paper E and F will be Hausdorff locally convex topological vector spaces and $T : E \to F$ a linear mapping. The domain of the adjoint operator, T', is defined to be

$$D(T') = \{y' \in F' : y'T \in E'\}$$

and

$$T' : D(T') \to E' \text{ is defined by } T'y' = y'T.$$

Theorem 1. *T' is sequentially continuous with respect to the relative $\sigma(F', F)$ topology on $D(T')$ and the topology on E' of uniform convergence on $\sigma(E, E')-$ $\mathcal{K}$ convergent sequences. In particular, T' is bounded with respect to these topologies.*

Proof. Let $\{y'_j\} \subseteq D(T')$ be $\sigma(F', F)$ convergent to 0 and let $\{x_i\} \subseteq E$ be a $\sigma(E, E') - \mathcal{K}$ sequence. Consider the matrix

$$M = [< T'y'_i, x_j >] = [< y'_i, Tx_j >].$$

It can be easily checked that M is a $\mathcal{K}$-matrix (see [1]), so by the Basic Matrix Theorem of [1], 2.1 or [6], 2.1

$$\lim_{i \to \infty} < T'y'_i, x_j >= 0$$

uniformly for $j \in \mathcal{N}$.

In particular, if E is a normed vector space, we obtain an interesting Corollary. First, we require

Lemma 1. *Let E be normed vector space. A subset B of X' is norm bounded iff B is uniformly bounded on sequences which are norm convergent to 0.*

Proof. It is obvious that a normed bounded subset B of X' is uniformly bounded on sequences which are norm convergent to 0. Suppose that the converse is not true, i.e. there exists a sequence $\{x'_n\}$ from B with $| x'_n |> n^2$. Pick $x_n \in X$ such that $| x_n |= 1$ and

$$|< x'_n, x_n >| > \{x'_n\} - \frac{1}{n}.$$

Then we have $| \frac{x_n}{n} |\to 0$ as $n \to \infty$. But

$$|< x'_n, \frac{x_n}{n} >| > n - \frac{1}{n^2},$$

is contradicting the hypothesis.

Corollary 1. *Let E be a normed $\mathcal{K}$ - space. Then T' carries weak* bounded subsets of $D(T')$ to norm bounded subsets of E'. In particular, T' is norm - bounded.*

Proof. In a normed $\mathcal{K}$ - space E, any sequence which converges to 0 is norm $\mathcal{K}$ convergent and so $\sigma(E, E')$ - $\mathcal{K}$ convergent. Hence by Theorem 1 and Lemma 1 we obtain the desired conclusion.

The boundedness version of the Adjoint Theorem for locally convex topological vector spaces established previously in [9] using the topology $\mathcal{K}(E', E)$ can be improved by using the topology $\beta^*(E', E)$, i.e. the topology on E' of uniform convergence on $\beta(E, E')$ bounded subsets of E. $\mathcal{K}(E', E)$ is the locally convex topology on E' of uniform convergence on the $\sigma(E, E')$ - $\mathcal{K}$ bounded subsets of E ([12]). The topology $\beta^*(E', E)$ is stronger than $\mathcal{K}(E', E)$ (see Lemma 3 and Remark 2) so the boundedness result below in Theorem 3 improves the earlier version.

Lemma 2. *If $A \subseteq E$ is absolutely convex, bounded and sequentially complete, then A is $\mathcal{K}$ - bounded.*

Proof. Let $\{x_j\}$ be a sequence from A and $\{t_j\}$ be a sequence of scalars such that $t_j \to 0$. Given any subsequence of $\{t_j\}$ pick a further subsequence $\{t_{n_j}\}$ such that

$$\sum_{j=1}^{\infty} | t_{n_j} | \leq 1.$$

Let p be a continuous semi-norm on E. The partial sums $S_k = \sum_j^k t_{n_j} x_{n_j}$ belong to A and form a Cauchy sequence in A since if $k > \ell$

$$p(S_k - S_\ell) \leq \sum_{j=\ell+1}^{k} | t_{n_j} | p(x_{n_j}) \leq \sup_{x \in A} p(x) \sum_{j=\ell+1}^{k} | t_{n_j} | .$$

Hence the series $\sum_{j=1}^{\infty} t_{n_j} x_{n_j}$ converges in A since A is sequentially complete.

Remark 1. Note that Lemma 2 apply if A is absolutely convex and compact.

Lemma 3. *If $B \subset E'$ is $\sigma(E', E)$ - $\mathcal{K}$ bounded, then B is $\beta(E', E)$ bounded.*

Proof. Let $A \subseteq E$ be $\sigma(E, E')$ bounded. It suffices to show that $\{< x'_i, x_i >\}$ is bounded whenever $\{x'_i\} \subseteq B$ and $\{x_i\} \subseteq A$. Let $\{t_i\}$ be a sequence of positive real numbers which tends to zero. Consider the matrix

$$M = [< \sqrt{t_j} x'_j, \sqrt{t_i} x_i >].$$

Since $\{\sqrt{t_i} x_i\}$ is $\sigma(E, E')$ convergent to 0 and $\{\sqrt{t_j} x'_j\}$ is $\sigma(E', E)$- $\mathcal{K}$ convergent, M is a K-matrix. Then by the Basic Matrix Theorem, $t_i < x'_i, x_i > \to 0$ as $i \to \infty$. Hence $\{< x'_i, x_i >\}$ is bounded.

Remark 2. Lemma 3 implies

$$\beta^*(E, E') \subseteq \mathcal{K}(E, E') \quad \text{and} \quad \mathcal{K}(E, E') \subseteq \tau(E, E')$$

(see [12]).

We shall need the following known result (since we could not find a good reference for it, we shall prove this theorem for the sake of completeness of the paper).

Theorem 2. *A subset B of E is $\sigma(E,E')$ bounded iff B is $\beta^*(E,E')$ bounded.*

Proof. It is clear that $\beta^*(E,E')$ boundedness implies boundedness since $\beta^*(E,E')$ is stronger than $\sigma(E,E')$. Now, suppose that $B \subseteq E$ is bounded. Let $A \subseteq E'$ be $\beta(E',E)$ (equivalently $\sigma(E',E''))$ bounded. Then A_0, the polar of A, is a basic $\beta^*(E,E')$ neighbourhood of 0 in E' and so there exists $t > 0$ such that $A \subseteq tB^0$. Then

$$A_0 \subseteq (\frac{1}{t})B_0^0 \subseteq (\frac{1}{t})B,$$

where B_0^0 is the bipolar of B.

Hence B is absorbed by A_0.

Remark 3. This theorem shows that any bounded set is $\mathcal{K}$(E,E') bounded and improved the result in [11].

We have the following boundedness result for the adjoint operator.

Theorem 3. *T' carries weak* bounded subsets of $D(T')$ into $\beta^*(E',E)$ bounded sets.*

Proof. Let $B \subseteq D(T')$ be weak* bounded. We claim $T'B$ is $\sigma(E',E)$ bounded. Let $x \in E$. Since B is weak* bounded we have

$$\sup_{y' \in B} |< T'y', x >| = \sup_{y' \in B} |< y', Tx >| < \infty.$$

By Theorem 2 the set $T'B$ is $\beta^*(E',E)$ bounded.

Remark 4. Theorem 3 improves Theorem 3 in [9].

If E is an $\mathcal{A}$ - space, i.e. every bounded subset of E is $\mathcal{K}$- bounded, then we have the following two results.

Theorem 4. *If E is an $\mathcal{A}$-space for any locally convex topology which is compatible with the duality between E and E', then every weakly bounded subset of E is strongly bounded and $\beta(E',E) = \beta^*(E',E)$.*

Proof. By [6] $(E, \sigma(E,E'))$ is an $\mathcal{A}$ - space. Hence by Lemma 3 any $\sigma(E,E')$ bounded set is $\beta(E,E')$ bounded and $\beta(E,E') = \beta^*(E,E')$.

In particular, if E i sequentially complete, then E is an $\mathcal{A}$ space (see [6]). So every weakly bounded subset of E is strongly bounded; this result is often referred to as the Banach-Mackey Theorem ([4], 20.1 (8), [14] 10.4.8). Thus, Theorem 4 extends the Banach-Mackey Theorem to $\mathcal{A}$ - spaces.

Corollary 2. *([9], Theorem 4.) If E is an $\mathcal{A}$ - space, then T' carries weak* bounded sets of $D(T')$ into $\beta(E', E)$ bounded sets.*

Proof. Follows by Theorems 3 and 4.
This result generalizes Corollary since the strong topology of the dual of a normed space is just the dual norm topology. From Theorem 3 we shall obtain a general continuity result which gives Theorem 5 from [9] as a special case.

Theorem 5. *If $D(T') = F'$ (equivalently, if T is $\sigma(E, E') - \sigma(F, F')$ continuous), then T' is $\beta^{**}(E, E') - \beta(F, F')$ continuous.*

Proof. Let $B \subseteq F'$ be $\sigma(F', F)$ bounded. Then TB is $\beta^*(E', E)$ bounded by Theorem 3. Since $(T'B)^0$ is the basic $\beta^{**}(E, E')$ neighbourhood and B_0 is the basic $\beta(F, F')$ neighbourhood, we have that the equality $(T'B)_0 = T^{-1}(B_0)$ gives the result.
From Theorems 5 and 4 we have

Corollary 3. *([9], Theorem 5). If E is an $\mathcal{A}$ - space and $D(T') = F'$, then T is $\beta^*(E, E') - \beta(F, F')$ continuous.*

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