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ON M_N SUBSETS

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Abstract

The aim of the present paper is to study some properties of M_N subsets and closed (almost closed) mappings.

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1. Introduction

No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is *regular open (regular closed)* iff $Int\ CLA = A$($CL\ Int\ A = A$), [6].

A subset A of a space X is *α-paracompact* (*α-nearly paracompact*) *with respect to a subset B* iff for every open (regular open) cover $\mathcal{U} = \{U_i : i \in I\}$ of A there is an open family $\mathcal{V} = \{V_j : j \in J\}$ such that:

1) $\mathcal{V}$ refines $\mathcal{U}$; 2) $A \subset \cup\{V_j : j \in J\}$ and

3) $\mathcal{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are *mutually α-paracompact (mutually α-nearly paracompact)* iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and B is α-paracompact (α-nearly paracompact) with respect to the subset A, [3].

A subset A of a space X is *α-Hausdorff* iff for any two points a, b of a space X, where $a \in A$ and $b \in X \backslash A$, there are disjoint open sets U and V containing a and b respectively.

A subset A of a space X is *α-regular* (α-almost regular) iff for any point $a \in A$ and any open (regular open) set U containing a, there is an open set V such that $a \in V \subset Cl\, V \subset U$ [4].

A proper subset A of a space X is a M_N *subset* iff:

a) $A \neq \emptyset$

b) A is α-Hausdorff α-nearly paracompact with respect to $X \backslash A$

c) Any two distinct points of a subset A cannot be strongly separated by open neighbourhoods [2].

A subset A of a space X is *α-nearly compact (N-closed)* iff for every regular open covering $\mathcal{U} = \{U_i : i \in J\}$ of A there is a finite subfamily I_0 of I such that $A \subset U\{U_i : i \in I_0\}$ [1].

A mapping $f : X \to Y$ is *closed (almost closed)* iff for every closed (regular closed) set F of X the set $f(F)$ is closed, [6].

2. M_N subsets

Lemma 1. *Let A and B be any two subsets of a space X. If $B \subset A$, $B \neq A$, and A is a M_N subset, then B is not an M_N-subset.*

Proof. B is not α-Hausdorff because any two distinct points $a \in B$, $b \in A \backslash B$ cannot be strongly separated by open neighbourhoods.

Lemma 2. *If A is an α-paracompact (α-nearly paracompact) subset with respect to a subset B and C is a closed (regular closed) subset of A, then C is α-paracompact (α-nearly paracompact) with respect to B.*

Proof. If $\mathcal{U} = \{U_i : i \in I\}$ is any open (regular open) covering of a subset C, then $\mathcal{H} = \mathcal{U} \cup \{X \backslash B\}$ is an open (a regular open) covering of A. Since A is α-paracompact (α-nearly paracompact) with respect to B it follows that there is an open family $\mathcal{V} = \{V_i : i \in J\}$ such that:

1. $A \subset \cup\{V_j : j \in J\} \cup \{X \backslash C\}$;
2. $\mathcal{V}$ refines $\mathcal{H}$;
3. $\mathcal{V}$ is locally finite at each point $x \in B$.

If $\mathcal{W} = \{V \in \mathcal{V} : V \cap C \neq \emptyset\}$, then $\mathcal{W}$ is an open family such that:

1. $C \subset \cup\{W : W \in \mathcal{W}\}$
2. $\mathcal{W}$ refines $\mathcal{U}$
3. $\mathcal{W}$ is locally finite at each point $x \in B$.

Hence it follows that C is α-paracompact (α-nearly paracompact) with respect to B.

In the paper [2] the author has proved that every M_N subset is closed. The converse statement is not necessarily true.

Theorem 1. *Let X be a topological space such that every proper nonempty closed subset is M_N. Then X can contain only two proper nonempty closed subsets.*

Proof. If A is only one proper closed subset, then A is not M_N because A is not α-Hausdorff. Let X be a space which contains at least three different proper nonempty closed subsets A, B, C.

a) If $A \subset B$ and B is a M_N subset, then the subset A is not M_N.

b) If $A \not\subset B$ and $B \not\subset A$, then $A \cup B$ is proper closed subset which is not M_N.

(The union $A \cup B$ of two different M_N subsets is not M_N because the points $a \in A$ and $b \in B$ can be strongly separated by open neighbourhoods).

The following example shows that there is a space such that any proper nonempty closed subset is M_N.

Example 1. Let $X = \{a, b, c, d\}$ and $C = \{\emptyset, \{a, b\}, \{c, d\}, X\}$.

The only two proper nonempty closed subsets are $A = \{a, b\}$ and $B = \{c, d\}$.

The subsets A and B are M_N.

Theorem 2. *Let Y be a space such that every proper nenempty closed subset is M_N. Then, any M_N subset is clo-open.*

Proof. The space contains only two disjoint closed M_N subsets A and B, hence result.

Theorem 3. *If A is an α-nearly paracompact subset with respect to a subset B of a space X and C is an α-nearly compact subset of a space Y, then $A \times C$ is an α-nearly paracompact subset with respect to $B \times Y$.*

Proof. Let $\mathcal{U}$ be any regular open covering of $A \times C$. Let $(x, y) \in A \times C$. There exist regular open subsets V_{xy} and W_{xy} of X and Y respectively such that $(x, y) \in V_{xy} \times W_{xy} \subset U$ for some $U \in \mathcal{U}$.

Let $I^x = \{x\} \times C$ for each $x \in A$. Then $\{W_{xy} : (x, y) \in I^x\}$ is a regular open covering of the α-nearly compact set C. Hence there is a finite subset J^x of I^x such that $\{W_{xy} : (x, y) \in J^x\}$ is a covering of C. For each $x \in A$, let $V_x = \cap\{V_{xy} : (x, y) \in J^x\}$. Let $\mathcal{V} = \{V_x : x \in A\}$. Then $\mathcal{V}$ is a regular open covering of A, hence there is a family $\mathcal{G}$ of open sets in X such that:

- $A \subset \cup\{G : G \in \mathcal{G}\}$;
- $\mathcal{G}$ refines $\mathcal{V}$;
- $\mathcal{G}$ is locally finite at each point $x \in B$.

Now, for each $G \in \mathcal{G}$, there is $x_G \in A$ such that $G \subset V_{x_G}$. Let $\mathcal{H} = \{GXW_{xy} : G \in \mathcal{G}, (x, y) \subset J^{x_G}\}$.

It is easy to verify that

- $A \backslash C \subset \cup\{H : H \in \mathcal{H}\}$;

- $\mathcal{H}$ refines $\mathcal{U}$;
- the open family $\mathcal{H}$ is locally finite at each point of $B \times Y$.

Hence $A \times C$ is α-nearly paracompact with respect to $B \times Y$.

Theorem 4. *Let X and Y be any two topological spaces. Let A be any M_N subset of X and $B \neq \emptyset$ by any proper subset of Y such that*

- *B is α-Hausdorff α-nearly compact*
- *any two points of B cannot be strongly separated by open neighbourhoods,*

then $A \times B$ is a M_N subset of a space $X \times Y$.

Proof. By the preceding theorem the subset $A \times B$ is α-nearly paracompact with respect to $X \times Y \backslash A \times B$. Since the product of two α-Hausdorff sets is α-Hausdorff, it follows that the set $A \times B$ is α-Hausdorff. Any two points $(a, b) \in A \times B$, $(c, d) \in A \times B$ can not be strongly separated by open neighbourhoods. Hence the set $A \times B$ is M_N.

Theorem 5. *Let X be a topological space such that any nonempty closed (regular closed) set F is an α-Hausdorff subset which is α-paracompact (α-nearly paracompact) with respect to $X \backslash F$. Then X is regular (almost regular).*

Proof. Let $x \notin F$ be any point. It follows that by Theorem 2.2. in [3], there are regular open sets U and V such that $x \in U$, $F \subset V$, $U \cap V = \emptyset$. It follows that X is regular (almost regular).

3. Almost closed mappings

Theorem 6. *Let A be an M_N subset of a topological space X and $f : X \to Y$ be a mapping of a space X onto a topological space Y.*

a) If there is $x \in X \backslash A$ such that $f^{-1}(f(x)) \cap A \neq \emptyset$ and f is an almost closed mapping such that the family $\{f^{-1}(f(x)) : x \in X \backslash A\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then $f(A)$ is closed.

b) If f is an almost closed mapping such that $f(A) \neq Y$ and for each $x \in X \backslash A$ $f^{-1}(f(x))$ is α-Hausdorff α-nearly paracompact with respect to $X \backslash f^{-1}(f(x))$, then $f(A)$ is closed.

c) If f is a closed mappping, then $f(A)$ is closed.

Proof.

a) Since there is $a \in X \backslash A$ such that $f^{-1}(f(a)) \cap A \neq \emptyset$ then $f(A) = a$, hence by Theorem 3.1. in [3] Y is Hausdorff. Hence $f(A)$ is closed.

b) It is similar to the proof. of a).

c) Since A is closed and f is a closed mapping, then $f(A)$ is closed.

Theorem 7. *Let A be an M_N subset of a topological space X and $f : X \to Y$ be a mapping of a space X onto a compact space Y.*

a) Let f be an almost closed mapping. If there is a point $x \in X \backslash A$ such that $f^{-1}(f(x)) \cap A \neq \emptyset$ and the family $\{f^{-1}(f(x)) : x \in X \backslash A\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact then $f(A)$ is compact.

b) If f is a closed mapping, then $f(A)$ is compact.

c) If f is an almost closed mapping such that $f(A) \neq Y$ and for each $x \in X \backslash A$ $f^{-1}(f(x))$ is α-Hausdorff α-nearly paracompact with respect to $X \backslash f^{-1}(f(x))$, then $f(A)$ is compact.

d) If f is an almost closed mapping such that the family $\{f^{-1}(f(x)) : x \in X \backslash A\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then f is continuous at each point $x \in X \backslash A$.

Proof. a); b); c) By the preceding theorem the set $f(A)$ is closed, hence $f(A)$ is compact.

Proof. $f(A)$ is compact, hence $f(A)$ is α-nearly paracompact with respect to $Y\backslash f(A) \cdot f(A)$ is α-Hausdorff. Any two points $a, b \in f(A)$ cannot by strongly separated by open neighbourhoods. Hence $f(A)$ is M_N.

In the preceding theorem the assumption "almost closed and continuous" cannot be replaced by "almost closed" which we can see from the following example.

Example 2. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a,b\}, \{c\}, \{d\}, \{a,b,c\}, \{a,b,d\}, \{c,d\}, X\}$.

Let $Y = \{1,2,3,4\}$ be endowed by discrete topology. Define the mapping $f : X \to Y$ by $f(a) = 1$; $f(b) = 2$; $f(c) = 3$; $f(d) = 4$. $A = \{a,b\}$ is a M_N subset of a space X.

$$f(A) = \{1,2\} \text{ is not } M_N.$$

(The points $1, 2 \in f(A)$ can be strongly separated by open neighbourhoods $\{1\}$ and $\{2\}$ respectively).

f is almost closed, f is not continuous at the points a and b, hence f is not continuous.

References

[1] Carnahan, D., Locally nearly compact spaces, Boll. Un. Mat. Ital. 4(6) (1972), 146-153.

[2] Kovačević, I., On almost closed mappings, paracompactness and partial equivalence relations, Indian J. Pure Appl. Math. 25(9) (1994), 949-954.

[3] Kovačević, I., On subsets, almost closed mappings and paracompactness, Glasnik matematički, 24(44) (1989), 125-132.

[4] Kovačević, I., Subsets and paracompactness, Univ. N. Sadu, Zb. Rad. Prir. Mat. Fak., 14(2) (1984), 79-87.

[5] Noivi, T., Almost continuity and some separation axioms, Glasnik matematički, 9(29) (1974), 131-135.

[6] Singal, M.K., Singal, A.R., Almost continuous mappings, Yokohoma Math. J., 16 (1968), 68-73.

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