NOVI SAD J. MATH.
VOL. 27, NO. 1, 1997, 53-69

DETERMINANTS OF RECTANGULAR MATRICES AND MOORE-PENROSE INVERSE

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Abstract

We investigate determinants of rectangular matrices and generalized inverses. Moreover, the correlation between induced generalized inverses, the Moore-Penrose inverse and the well-known determinantal representation of the Moore-Penrose inverse is considered.

AMS Mathematics Subject Classification (1991): 15A09, 15A60
Key words and phrases: Moore-Penrose inverse, determinants of rectangular matrices, generalized inverses, determinantal representation, generalized algebraic complement

1. Introduction

Let C^n be the n-dimensional complex vector space, $C^{m\times n}$ the set of $m \times n$ complex matrices, and $C_r^{m\times n} = \{X \in C^{m\times n} : rank(X) = r\}$. The adjungate matrix of a square matrix B is denoted by $adj(B)$, and its determinant is denoted $det(B)$. Conjugate, transposed and conjugate-transposed matrix of A is denoted by $\overline{A}$, A^T and A^*, respectively. Minor of A containing rows $\alpha_1, \ldots, \alpha_t$ and columns $\beta_1, \ldots, \beta_t$ is denoted by $A\begin{pmatrix} \alpha_1 & \ldots & \alpha_t \\ \beta_1 & \ldots & \beta_t \end{pmatrix}$, and

$$A_{ij}\begin{pmatrix} \alpha_1 & \dots & \alpha_{p-1} & i & \alpha_{p+1} & \dots & \alpha_t \\ \beta_1 & \dots & \beta_{q-1} & j & \beta_{q+1} & \dots & \beta_t \end{pmatrix} =$$
$$(-1)^{p+q}A\begin{pmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_t \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_t \end{pmatrix}.$$

Recall that for $A \in C^{m\times n}$ there exists a unique matrix $A^{+} = X \in C^{n\times m}$ such that

$$AXA = A \tag{1.1}$$
$$XAX = X \tag{1.2}$$
$$(AX)^* = AX \tag{1.3}$$
$$(XA)^* = XA, \tag{1.4}$$

known as the *Moore-Penrose inverse* of A [7]. A matrix satisfying the condition (1.1) is called a generalized inverse of A and is denoted by $A^{(1)}$. A matrix which satisfies conditions (1.1) and (1.2) is called a reflexive generalized inverse of A, and is denoted by $A^{(1,2)}$. A matrix satisfying the conditions (1.1), (1.2) and (1.3) is called a right (left) normalized generalized inverse of A, and is denoted by $A^{(1,2,3)}$. Similarly, a matrix satysfying the conditions (1.1), (1.2) and (1.4) is called a left normalized generalized inverse of A, and is denoted by $A^{(1,2,4)}$.

The set of matrices satisfying the conditions $(1.i), (1.j), \dots, (1.l)$ is denoted by $A\{i, j, \dots, l\}$. A matrix $A \in C^{m\times n}$ is said to be left (respectively, right) invertible if there exists a matrix A_l^{-1} (respectively A_r^{-1}) from $C^{n\times m}$ such that $A_l^{-1}A = I_n$ (respectively, $AA_r^{-1} = I_m$). A matrix A_l^{-1} (respectively A_r^{-1}) satisfying this condition is called left (respectively, right) inverse of A. (I_m denotes a unit matrix of the order $m \times m$).

Theorem 1.1. [1] *Let $A \in C^{m\times n}$ be a full-rank matrix. If $rank(A) = m \le n$ the system*

$$AX = I_m; \tag{1.5}$$
$$(XA)^* = XA \tag{1.6}$$

has a unique solution $X = A^{+}$. Similarly, if $m > n = rank(A)$, then the following system has a unique solution $X = A^{+}$:

$$XA = I_n; \tag{1.7}$$
$$(AX)^* = AX. \tag{1.8}$$

Determinantal representation of the *Moore-Penrose inverse* is studied in [1], [2], [3], [4], [6]. The main result of these papers is:

Theorem 1.2. [2], [3], [4], [6] *Element* $a_{ij}^{(+,r)}$, $\begin{pmatrix} 1 \le i \le n \\ 1 \le j \le m \end{pmatrix}$ lying on the i-row and j-column of the Moore-Penrose pseudoinverse of a given matrix $A \in C_r^{m\times n}$ is given by

$$a_{ij}^{(+,r)} = \frac{A_{ji}^{(+,r)}}{N_r(A)} = \frac{\sum\limits_{\substack{1\le\beta_1<\dots<\beta_r\le n\\ 1\le\alpha_1<\dots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\delta_1<\dots<\delta_r\le n\\ 1\le\gamma_1<\dots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}\overline{A}\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}.$$

We denote by $adj^{(+,r)}(A)$ matrix whose (i,j)th element is $A_{ji}^{(+,r)}$.

General forms of generalized inverses are described in the following theorem.

Theorem 1.3. [8] *If* $A \in C_r^{m\times n}$ *has a full-rank factorization* $A = PQ$, $P \in C_r^{m\times r}$, $Q \in C_r^{r\times n}$, $W_1 \in C^{n\times r}$ *and* $W_2 \in C^{r\times m}$ *are some matrices such that* $rank(QW_1) = rank(W_2P) = rank(A)$, *then*

$$\begin{aligned}
A^+ &= Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^+P^+\\
A\{1,2\} &= \{W_1(QW_1)^{-1}(W_2P)^{-1}W_2 = Q_r^{-1}P_l^{-1}\}\\
A\{1,2,3\} &= \{W_1(QW_1)^{-1}(P^*P)^{-1}P^* = Q_r^{-1}P^+\}\\
A\{1,2,4\} &= \{Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2 = Q^+P_l^{-1}\}.
\end{aligned}$$

The notion of *determinants of rectangular matrices* has been introduced in [9], [10], [11] by M. Stojaković and M. Radić. Their definitions are contained in the following definition:

Definition 1.1. *Determinat of* $A \in C_r^{m\times n}$ *is a function* $det_{(\epsilon,p)} : C^{m\times n} \longrightarrow C$ *defined by:*

$$det_{(\epsilon,p)}(A) = \sum_{\alpha_1<\dots<\alpha_p\beta_1<\dots<\beta_p} \epsilon^{(\alpha_1+\dots+\alpha_p)+(\beta_1+\dots+\beta_p)} A\begin{pmatrix}\alpha_1 & \dots & \alpha_p\\ \beta_1 & \dots & \beta_p\end{pmatrix}, \textit{for}$$

$p \le \min\{m,n\}$, *and* $det_{(\epsilon,p)}(A) = 0$, *otherwise.*

For $\epsilon = 1$ we get the Stojaković determinant, denoted by $det_{(S,p)}(A)$. Similarly, for $\epsilon = -1$, we get the determinant introduced by M. Radić (it is denoted by $det_{(R,p)}(A)$).

Later, in [5], V.N. Joshi has defined a determinant of rectangular full-rank matrices, as follows.

Definition 1.2. *Let m, $p_1, \ldots, p_m$ be integers which satisfy the following conditions:*

$$\begin{array}{ll} (i) & m \leq n; \\ (ii) & p_i \in \{1, \ldots, n\} \quad \text{for all} \quad i \in \{1, \ldots, m\}; \\ (iii) & p_1 < \ldots < p_m. \end{array}$$

For an integer d, $1 \leq d \leq (n - m + 1)$ consider the set

$$S_d = \{e_{d,p} = (d, p_2, \ldots, p_m) \mid d < p_2 < \ldots < p_m \leq n\}.$$

For a rectangular matrix $A \in C^{m \times n}$ $(m \leq n)$ let $A_{d,p}$ be $m \times m$ submatrix whose columns conform to the ordering of integers in

$$e_{d,p}, \quad 1 \leq d \leq n - m + 1, \quad 1 \leq p \leq N_d = \binom{n-d}{m-1}.$$

Determinant of A is the number

$$det_{(J,m)}(A) = \sum_{d=1}^{n-m+1} \sum_{p=1}^{N_d} det\,(A_{d,p}).$$

For an $m \times n$ matrix $(m > n)$, $det_{(J,n)}(A)$ is equal to $det_{(J,n)}(A^T)$.

2. Determinants of rectangular matrices

In this section we investigate connections between the presented definitions of the *rectangular determinants* and their main properties.

Theorem 2.1. *For $A \in C_r^{m \times n}$, $r = \min\{m, n\}$ is valid*

$$det_{(J,r)}(A) = det_{(S,r)}(A).$$

Proof. Suppose that $m \leq n$. By (ii) and (iii) we get $e_{d,p}$ are the combinations of m elements selected from the set $\{1, \ldots, n\}$ which start with the number $d = p_1$. Thus,

$$det(A_{d,p}) = A\begin{pmatrix} 1 & \ldots & \ldots & m \\ d & p_2 & \ldots & p_m \end{pmatrix} = A\begin{pmatrix} 1 & \ldots & m \\ p_1 & \ldots & p_m \end{pmatrix}, \quad p_1 = d <$$

$p_2 < \ldots < p_m \leq n, \quad 1 \leq d \leq n - m + 1.$

Hence,

$$\sum p = 1^{N_d} det\left(A_{d,p}\right) = \sum_{d=p_1<\ldots<p_m\leq n} A\begin{pmatrix} 1 & \ldots & m \\ p_1 & \ldots & p_m \end{pmatrix}, \quad 1 \leq d \leq n-m+1,$$

and finally,

$$det_{(J,m)}(A) = \sum_{d=1}^{n-m+1} \sum_{p=1}^{N_d} det\left(A_{d,p}\right) = \sum_{1\leq q_1<\ldots<q_m\leq n} A\begin{pmatrix} 1 & \ldots & m \\ q_1 & \ldots & q_m \end{pmatrix} = det_{(S,m)}(A).$$

□

Corollary 2.1. *If* $A \in C_r^{m\times n}$*, then*

$$det_{(J,p)}(A) = det_{(R,p)}\left(A^{\odot}\right), \quad \text{where} \quad A^{\odot} = \left[(-1)^{i+j} a_{ij}\right] \quad \text{and} \quad p \leq r.$$

Proof. The proof is an easy consequence of Theorem 2.1. and the following relation, which is proved in [9], [10]: $det_{(R,p)}(A) = det_{(S,p)}\left(A^{\odot}\right)$. □

In [5], [9], [10], [11] are presented some important properties of *rectangular determinants* for full-rank matrices. The following lemma is valid for an arbitrary matrix

Lemma 2.1. *For* $A \in C_r^{m\times n}$ *and* $p \leq r$ *is valid:*

a) $det_{(\epsilon,p)}(cA) = c^p det_{(\epsilon,p)}(A)$, $c \in C$. *b)* $det_{(\epsilon,p)}(A^*) = \overline{det_{(\epsilon,p)}(A)}$.

The multiplicative property of rectangular determinants is proved in [11]. A similar claim, presented in [10] is not valid. We prove this property using an original proof.

Lemma 2.2. *For $A \in C_r^{m\times r}$, $B \in C_r^{r\times n}$ and $r \le \min\{m,n\}$ the following relation can be proved*

$$det_{(\epsilon,r)}(AB) = det_{(\epsilon,r)}(A)det_{(\epsilon,r)}(B).$$

Proof. According to Definition 1.1 we obtain

$$det_{(\epsilon,r)}(AB) = \sum_{\substack{1\le j_1<\ldots<j_r\le n\\ 1\le i_1<\ldots<i_r\le m}} \epsilon^{(i_1+\ldots+i_r)+(j_1+\ldots+j_r)}(AB)\begin{pmatrix} i_1 & \ldots & i_r\\ j_1 & \ldots & j_r\end{pmatrix} =$$

$$= \sum_{\substack{1\le j_1<\ldots<j_r\le n\\ 1\le i_1<\ldots<i_r\le m}} \epsilon^{(i_1+\ldots+i_r)+(j_1+\ldots+j_r)} A\begin{pmatrix} i_1 & \ldots & i_r\\ 1 & \ldots & r\end{pmatrix} B\begin{pmatrix} 1 & \ldots & r\\ j_1 & \ldots & j_r\end{pmatrix} =$$

$$= \left[\sum_{1\le i_1<\ldots<i_r\le m} \epsilon^{(i_1+\ldots+i_r)+(1+\ldots+r)} A\begin{pmatrix} i_1 & \ldots & i_r\\ 1 & \ldots & r\end{pmatrix}\right]\cdot$$

$$\cdot\left[\sum_{1\le j_1<\ldots<j_r\le n} \epsilon^{(1+\ldots+r)+(j_1+\ldots j_r)} B\begin{pmatrix} 1 & \ldots & r\\ j_1 & \ldots & j_r\end{pmatrix}\right] = det_{(\epsilon,r)}(A)\cdot det_{(\epsilon,r)}(B).$$

□

In the following example is shown the existence of the matrices A and B such that $det_{(\epsilon,p)}(AB) \neq det_{(\epsilon,p)}(A)det_{(\epsilon,p)}(B)$, for some $p < r = rank(A) = rank(B)$.

Example 2.1. Consider matrices $A = \begin{pmatrix} 2 & -\frac{1}{3} & \frac{5}{4} & 2\\ 0 & \frac{7}{5} & -1 & \frac{14}{3}\\ 0 & 1 & -\frac{5}{2} & \frac{3}{7}\end{pmatrix}$ and $B = \begin{pmatrix} -11 & \frac{4}{3} & 0\\ 0 & 1 & 2\\ -2 & \frac{2}{5} & -\frac{1}{5}\\ 5 & \frac{3}{5} & 0\end{pmatrix}$.

Then, $det_{(S,2)}(AB) = \frac{111259}{252} \neq \frac{3186341}{3150} = det_{(S,2)}(A)det_{(S,2)}(B)$.

In [5], [10], [11] is developed a generalization of the cofactor expansion. We prove the same theorem using a short proof.

Theorem 2.2. *For a full-rank matrix $A \in C^{m\times n}$ is valid Laplace's development:*

$$\begin{cases} det_{(\epsilon,m)}(A) = \sum\limits_{k=1}^{n} a_{ik}A_{ki}^{(\epsilon,m)}, & i=1,\ldots,m, \quad m\leq n \\ det_{(\epsilon,n)}(A) = \sum\limits_{k=1}^{m} a_{ik}A_{ki}^{(\epsilon,n)}, & i=1,\ldots,n, \quad n\leq m \end{cases},$$

where $A_{ij}^{(\epsilon,m)}$, i.e. $A_{ij}^{(\epsilon,n)}$ is the generalized algebraic complement corresponding to the element a_{ji}, defined as follows

$$\begin{cases} A_{ij}^{(\epsilon,m)} = \sum\limits_{j_1<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)} A_{ji}\begin{pmatrix} 1 & \ldots & i & \ldots & m \\ j_1 & \ldots & j & \ldots & j_m \end{pmatrix}, & m\leq n \\ A_{ij}^{(\epsilon,n)} = \sum\limits_{i_1<\ldots<i_n} \epsilon^{(i_1+\ldots+i_n)+(1+\ldots+n)} A_{ji}\begin{pmatrix} i_1 & \ldots & i & \ldots & i_n \\ 1 & \ldots & j & \ldots & n \end{pmatrix}, & n\leq m. \end{cases}$$

Proof. In the case $m\leq n$, according to Definition 1.1. and using Laplace's development for the square minors $A\begin{pmatrix} 1 & \ldots & m \\ j_1 & \ldots & j_m \end{pmatrix}$, we get:

$$\begin{aligned} det_{(\epsilon,m)}(A) &= \\ &= \sum_{j_1<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)} \left[\sum_{k=1}^{m} a_{ij_k} A_{ij_k}\begin{pmatrix} 1 & \ldots & i & \ldots & m \\ j_1 & \ldots & j_k & \ldots & j_m \end{pmatrix}\right] \\ &= \sum_{l=1}^{n} a_{il}\left[\sum_{1\leq j_1<\ldots<j_m\leq n} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)} A_{il}\begin{pmatrix} 1 & \ldots & i & \ldots & m \\ j_1 & \ldots & l & \ldots & j_m \end{pmatrix}\right] \\ &= \sum_{l=1}^{n} a_{il}A_{li}^{(\epsilon,m)}. \end{aligned}$$

□

Corollary 2.2. *If $A \in C^{m\times n}$ is a full-rank matrix, then:*

$$\begin{cases} \sum\limits_{k=1}^{n} a_{ik}A_{kj}^{(\epsilon,m)} = \delta_{ij} det_{\epsilon,m}(A), & m\leq n \\ \sum\limits_{k=1}^{m} a_{ik}A_{kj}^{(\epsilon,n)} = \delta_{ij} det_{\epsilon,n}(A), & n\leq m \end{cases}, \quad where \quad \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}.$$

Proof. For $i = j$ we obtain the claim of Theorem 2.2. In the case $i \neq j$, starting from a matrix A whose ith and jth rows are identical, $1 \leq i \leq m$, $1 \leq j \leq n$, using Laplace's development on the jth row for the obtained square minors and the well-known fact: *rectangular determinant* of a full-rank matrix which has two identical rows is equal to zero, ([5], [9], [10], [11]), we obtain the proof. □

3. Rectangular determinants and induced generalized inverses

Now we present a definition of generalized inverses in terms of the *rectangular determinants* and *generalized cofactors*, which we call by *determinantal generalized inverse* .

Definition 3.1. *For $A \in C_r^{m\times n}$ generalized inverse $A^{-1}_{(\epsilon,p)}$ of A is the matrix whose (i,j)th entry is equal to*

$$\left(A^{-1}_{(\epsilon,p)}\right) = \frac{A_{ij}^{(\epsilon,p)}}{det_{(\epsilon,p)}(A)},$$

where $1 \leq p \leq rank(A) \leq \min\{m,n\}$ is the greatest integer, such that $det_p^{\epsilon}(A) \neq 0$ (denoted by $r_\epsilon(A)$), and $A_{ij}^{(\epsilon,p)}$ is the generalized algebraic complement of the order p corresponding to the element a_{ji}, defined as follows:

$$A_{ij}^{(\epsilon,p)} = \sum_{\substack{1\leq j_1<\dots<j<\dots<j_p\leq s \\ 1\leq i_1<\dots<i<\dots<i_p\leq r}} \epsilon^{(i_1+\dots+i_p)+(j_1+\dots+j_p)} A_{ji}\begin{pmatrix} j_1 & \dots & j & \dots & j_p \\ i_1 & \dots & i & \dots & i_p \end{pmatrix}.$$

The matrix $adj^{(\epsilon,p)}(A) = \left(A_{ij}^{(\epsilon,p)}\right), \begin{pmatrix} 1\leq i\leq n \\ 1\leq j\leq m\end{pmatrix}$ we shall call *generalized adjoint matrix of A of the order p.*

In the case $p = r_\epsilon(A) = r$ we obtain the corresponding notions of generalized inverses, introduced in [5], [10], [11]. Moreover, we investigate the properties of the introduced generalized inverses. The following theorem is proved in [5]. The proof is evident from Corollary 2.2.

Theorem 3.1. *If $p = r_\epsilon(A) = min\{m,n\}$ matrix $A^{-1}_{(\epsilon,p)}$ computed according to Definition 1.1 is a right inverse of A if $m < n$ and a left inverse in the case $m > n$.*

In the following two lemmas we examine the properties of *generalized adjoint matrices* and *determinantal inverses* .

Lemma 3.1. *If $A \in C^{m\times r}$, $B \in C^{r\times n}$ are two full rank matrices such that $rank(A) = r = rank(B) = r_\epsilon(A) = r_\epsilon(B) = r_\epsilon(AB)$, then $adj^{(\epsilon,r)}(AB) = adj^{(\epsilon,r)}(B)\cdot adj^{(\epsilon,r)}(A)$.*

Proof. An element lying in the ith row and jth column of $adj^{(\epsilon,r)}(AB)$ is equal to

$$(AB)^{(\epsilon,r)}_{ij} = \sum_{\substack{1\le\beta_1<\ldots<i<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<j<\ldots<\alpha_r\le m}} (AB)_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}.$$

Using the Cauchy-Binet formula, we can show

$$\begin{aligned}(AB)^{(\epsilon,r)}_{ij} &= \sum_{\substack{\beta_1<\ldots<i<\ldots<\beta_r\\ \alpha_1<\ldots<j<\ldots<\alpha_r}}\left[\sum_{k=1}^{r} A_{jk}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix}\cdot\right.\\ &\qquad\left.\cdot B_{ki}\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}\right] =\\ &= \sum_{k=1}^{r}\left[\sum_{1\le\beta_1<\ldots<i<\ldots<\beta_r\le n} B_{ki}\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}\right]\times\\ &\quad\times\left[\sum_{1\le\alpha_1<\ldots<j<\ldots<\alpha_r\le m} A_{jk}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix}\right] =\\ &= \sum_{k=1}^{r} B^{(\epsilon,r)}_{ik}A^{(\epsilon,r)}_{kj}. \qquad \square\end{aligned}$$

Lemma 3.2. *If $k = r_\epsilon(A)$, then the following equations are valid:*

a) $adj^{(\epsilon,k)}(cA) = c^{k-1}adj^{(\epsilon,k)}(A),\ c \in C;$

b) $adj^{(\epsilon,k)}(A^*) = \left(adj^{(\epsilon,k)}(A)\right)^*;$

c) $(cA)^{-1}_{(\epsilon,k)} = \frac{1}{c}A^{-1}_{(\epsilon,k)},\ c \in C;$

d) If $k = min\{m, n\}$ then

$$\left[det_{(\epsilon,m)}(A)\right]^m = \sum_{p_1<\ldots<p_m} A\begin{pmatrix} p_1 & \ldots & p_m \\ 1 & \ldots & m \end{pmatrix}\left(adj^{(\epsilon,m)}(A)\right)\begin{pmatrix} p_1 & \ldots & p_m \\ 1 & \ldots & m \end{pmatrix}, \quad m \leq n,$$

$$\left[det_{(\epsilon,n)}(A)\right]^n = \sum_{p_1<\ldots<p_n} A\begin{pmatrix} p_1 & \ldots & p_n \\ 1 & \ldots & n \end{pmatrix}\left(adj^{(\epsilon,n)}(A)\right)\begin{pmatrix} 1 & \ldots & n \\ p_1 & \ldots & p_n \end{pmatrix}, \quad n \leq m.$$

Proof. d) For $m \leq n$ matrix $A^{-1}_{(\epsilon,m)}$ is a right inverse of A, so that $A \cdot adj^{(\epsilon,m)}(A) = det_{(\epsilon,m)}(A) \cdot I_m$. Thus, $det\left(A \cdot adj^{(\epsilon,m)}(A)\right) = \left[det_{(\epsilon,m)}(A)\right]^m$. Applying the Cauchy-Binet Theorem, we get

$$\left[det_{(\epsilon,m)}(A)\right]^m = \sum_{p_1<\ldots<p_m} A\begin{pmatrix} 1 & \ldots & m \\ p_1 & \ldots & p_m \end{pmatrix}\left[\left(adj^{(\epsilon,m)}(A)\right)\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & m \end{pmatrix}\right].$$

□

If A is a square $m \times m$ matrix, we obtain the well-known result $det\,(adj(A)) = det^{m-1}(A)$.

According to Lemma 3.1 and Lemma 2.2, we can compute the *determinantal inverses* using the notion of the full-rank factorization.

Corollary 3.1. *If $A = PQ$ is a full-rank factorization of $A \in C_r^{m\times n}$, determinantal inverse of A is*

$$A^{-1}_{(\epsilon,r)} = Q^{-1}_{(\epsilon,r)}P^{-1}_{(\epsilon,r)}, \qquad r = r_\epsilon(A).$$

Using Theorem 1.3, we can immediately prove the following corollary.

Corollary 3.2. *If $A \in C_r^{m\times n}$, $r = r_\epsilon(A)$, then $A^{-1}_{(\epsilon,r)} = Q^{-1}_{(\epsilon,r)}P^{-1}_{(\epsilon,r)}$ is:*

- *The Moore-Penrose inverse of A if $Q^{-1}_{(\epsilon,r)} = Q^+$ and $P^{-1}_{(\epsilon,r)} = P^+$;*
- *Right normalized generalized inverse of A if $P^{-1}_{(\epsilon,r)} = P^+$, $Q^{-1}_{(\epsilon,r)} \neq Q^+$;*
- *Left normalized generalized inverse of A if $Q^{-1}_{(\epsilon,r)} = Q^+$, $P^{-1}_{(\epsilon,r)} \neq P^+$;*
- *Reflexive generalized inverse of A, in other cases.*

Example 3.1 Let $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. We have $m = 4$; $n = 3$; $rank(A) = 2 < \min\{m, n\}$. Using Definition 3.1, we obtain:

$$A^{-1}_{(S,2)} = \frac{1}{det_2^S(A)} \cdot$$

$$\cdot \begin{pmatrix} det_1^S \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 0 & -2 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 0 & -2 \\ -1 & -2 \\ 1 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 0 & -2 \\ -1 & -2 \\ -1 & -3 \end{pmatrix} \\ det_1^S \begin{pmatrix} 0 & 2 \\ -1 & 3 \\ 0 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & -2 \\ -1 & 3 \\ 0 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & -2 \\ 0 & -2 \\ 0 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & -2 \\ 0 & -2 \\ 1 & -3 \end{pmatrix} \\ det_1^S \begin{pmatrix} 0 & -1 \\ -1 & -1 \\ 0 & -1 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & -2 \\ -1 & 3 \\ 0 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & -2 \\ 0 & -2 \\ 0 & 2 \end{pmatrix} & det_1^S \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix} =$$

$$= \frac{1}{12} \begin{pmatrix} 10 & 5 & -2 & -9 \\ 6 & 4 & 0 & -4 \\ -4 & 1 & 2 & 5 \end{pmatrix}.$$

Similarly, from Definition 3.2 we get $A = PQ$; $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$;

$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

The right inverse of Q is

$$Q^{-1}_{(S,2)} = \frac{1}{det_2^S\,(Q)} \begin{pmatrix} det_1^S\,(\ 1 \quad 2\ ) & det_1^S\,(\ 0 \quad -1\ ) \\ det_1^S\,(\ 0 \quad 2\ ) & det_1^S\,(\ -1 \quad 1\ ) \\ det_1^S\,(\ 0 \quad -1\ ) & det_1^S\,(\ 1 \quad 0\ ) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

The left inverse of P is

$$P^{-1}_{(S,2)} = \frac{1}{det_2^S\,(P)} \cdot$$

$$\cdot \begin{pmatrix} det_1^S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & det_1^S \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & det_1^S \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} & det_1^S \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \\ det_1^S \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} & det_1^S \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} & det_1^S \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} & det_1^S \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} =$$

$$= \frac{1}{6} \begin{pmatrix} 3 & 2 & 0 & -2 \\ -1 & 1 & 2 & 3 \end{pmatrix},$$

and the generalized inverse of A is equal to

$$Q_{(S,2)}^{-1} \cdot P_{(S,2)}^{-1} = \frac{1}{12} \begin{pmatrix} 10 & 5 & -2 & -9 \\ 6 & 4 & 0 & -4 \\ -4 & 1 & 2 & 5 \end{pmatrix}.$$

Now we study the correlations between $A_{(R,k)}^{-1}$ and $A_{(S,k)}^{-1}$, $k=r_\epsilon(A)$.

Theorem 3.2. *For $A \in C_r^{m\times n}$ the following relations between Radić's and Stojaković's inverse can be proved.*

a) $A^{\odot}{}_{(R,k)}^{-1} = \left(A_{(S,k)}^{-1}\right)^{\odot}$; *b)* $A_{(S,k)}^{-1} = \left(A^{\odot}{}_{(R,k)}^{-1}\right)^{\odot}$;

c) $A^{\odot}{}_{(S,k)}^{-1} = \left(A_{(R,k)}^{-1}\right)^{\odot}$; *d)* $A_{(R,k)}^{-1} = \left(A^{\odot}{}_{(S,k)}^{-1}\right)^{\odot}$.

Proof. a) Element lying in the jth row and ith column of $A^{\odot}{}_{(R,k)}^{-1}$ is equal to

$$\left(A^{\odot}{}_{(R,k)}^{-1}\right)_{ji} = \frac{\sum\limits_{\substack{i_1<\dots<i_k \\ j_1<\dots<j_k}} (-1)^{(i_1+\dots+i_k)+(j_1+\dots+j_k)} A^{\odot}{}_{ij} \begin{pmatrix} i_1 & \dots & i & \dots & i_k \\ j_1 & \dots & j & \dots & j_k \end{pmatrix}}{det_{(S,k)}(A)} =$$

$$= \frac{(-1)^{i+j} \sum\limits_{\substack{i_1<\dots<i_k \\ j_1<\dots<j_k}} A_{ij} \begin{pmatrix} i_1 & \dots & i & \dots & i_k \\ j_1 & \dots & j & \dots & j_k \end{pmatrix}}{det_{(S,k)}(A)} = (-1)^{i+j} \frac{A_{ij}^{(S,k)}}{det_{(S,k)}(A)}$$

$$= (-1)^{i+j} \left(A_{(S,k)}^{-1}\right)_{ji}. \square$$

4. Rectangular determinants and Moore-Penrose inverse

Now, we investigate the correlation between the *determinantal generalized inverses* and the Moore-Penrose inverse.

Theorem 4.1. *For a rectangular full-rank matrix $A \in C_r^{m\times n}$, and $r = r_\epsilon(A) = \min\{m,n\}$, the relation $A^{-1}_{(\epsilon,r)} = A^+$ holds if and only if the matrix A satisfies one of the following two conditions:*

$$\frac{\sum\limits_{j_1<\ldots<p<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)}\overline{A}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & p & \ldots & j_m \end{pmatrix}}{\left(\overline{det_{(\epsilon,m)}(A)}\right)^{-1}} =$$

$$= \frac{\sum\limits_{j_1<\ldots<q<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)}A\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & q & \ldots & j_m \end{pmatrix}}{\left(det_{(\epsilon,m)}(A)\right)^{-1}};$$

(1)

$$\frac{\sum\limits_{j_1<\ldots<p<\ldots<j_{n-1}} \epsilon^{(1+\ldots+n)+(j_1+\ldots+j_n)}\overline{A}\begin{pmatrix} j_1 & \ldots & p & \ldots & j_n \\ 1 & \ldots & \ldots & \ldots & n \end{pmatrix}}{\left(\overline{det_{(\epsilon,n)}(A)}\right)^{-1}} =$$

$$= \frac{\sum\limits_{j_1<\ldots<q<\ldots<j_{n-1}} \epsilon^{(1+\ldots+n)+(j_1+\ldots+j_n)}A\begin{pmatrix} j_1 & \ldots & q & \ldots & j_n \\ 1 & \ldots & \ldots & \ldots & n \end{pmatrix}}{\left(det_{(\epsilon,n)}(A)\right)^{-1}}.$$

Proof. According to Theorem 3.1 and Theorem 1.1, $A^{-1}_{(\epsilon,m)} \neq A^+$ if and only if $A^{-1}_{(\epsilon,m)}A \neq \left(A^{-1}_{(\epsilon,m)}A\right)^*$. Indeed, the element γ_{pq} in the pth row and qth column of the matrix product $A^{-1}_{(\epsilon,m)} \cdot A$ is equal to

$$\frac{\sum\limits_{k=1}^{m} A^{(\epsilon,m)}_{pk} a_{kq}}{det_{(\epsilon,m)}(A)} = \frac{\sum\limits_{j_1<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+j_m)}\left[\sum\limits_{k=1}^{m} a_{kq}A_{kp}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & p & \ldots & j_m \end{pmatrix}\right]}{det_{(\epsilon,m)}(A)} =$$

$$= \frac{\sum\limits_{1\le j_1<\ldots<q<\ldots<j_m\le n} \epsilon^{(1+\ldots+m)+(j_1+\ldots+p+\ldots+j_m)}A\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & q & \ldots & j_m \end{pmatrix}}{det_{(\epsilon,m)}(A)}.$$

In a similar way it can be proved

$$\overline{\gamma}_{qp} = \frac{\sum\limits_{j_1<\ldots<p<\ldots<j_m} \epsilon^{(1+\ldots+m)+(j_1+\ldots+q+\ldots+j_m)}\overline{A}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & m \\ j_1 & \ldots & p & \ldots & j_m \end{pmatrix}}{\overline{det_{(\epsilon,m)}(A)}}. \qquad \square$$

In Theorem 4.1 we found a necessary and sufficient condition for detection of the equivalence of the *determinantal inverse* and the *Moore-Penrose inverse*. This condition is obtained applying Theorem 1.1. Moreover, in the following lemma, using determinantal representation of the *Moore-Penrose inverse* presented in Theorem 1.2, we find a sufficient condition for the equivalence of the *determinantal inverse* and the *Moore-Penrose inverse* .

Lemma 4.1. *If $r = r_\epsilon(A)$ and the matrix A satisfies the condition*

$$(2) \qquad \overline{A}\begin{pmatrix} i_1 & \dots & i_r \\ j_1 & \dots & j_r \end{pmatrix} = K \cdot \epsilon^{(i_1+\dots+i_r)+(j_1+\dots+j_r)}, \quad K \in C$$

$$\textit{for all combinations} \quad \begin{pmatrix} 1 \le j_1 < \dots < j_r \le n, \\ 1 \le i_1 < \dots < i_r \le m \end{pmatrix}$$

then $A^{-1}_{(\epsilon,r)} = A^+$.

Proof. For the chosen integers $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ it is trivial to verify that $N_r(A) = K \cdot det^\epsilon_r(A)$ and $A^{(+,r)}_{ji} = K \cdot A^{(\epsilon,r)}_{ji}$. □

The class of matrices satisfying conditions (2) is nonempty.

Example 4.1. The following matrices satisfy condition (2):

$$\begin{pmatrix} A & \epsilon^{1+2}(A+C) & C \\ \epsilon^{2+1}D & \frac{K+D(A+C)}{A} & \epsilon^{2+3}\frac{K+CD}{A} \end{pmatrix}, \quad A, B, C, D \in C, \quad \epsilon \in \{-1, 1\}.$$

Problem 4.1. Find a complex matrix A which does not satisfy relation (2), but $A^{-1}_{(\epsilon,r)} = A^+$.

According to Lemma 4.1 and Corollary 3.2 we describe an algorithm which allows detection of the type of *determinantal inverse* $A^{-1}_{(\epsilon,r_\epsilon(A))}$.

Algorithm 1.

Case 1. If $p = r_\epsilon(A) = min\{m,n\}$, then apply rules 1.1 and 1.2.

Rule 1.1 If A satisfies condition (2), then $A^{-1}_{(\epsilon,p)} = A^+$.

Rule 1.2 If condition (2) does not holds for A, then

a) For $m \leq n$, if $\left(A_{(\epsilon,p)}^{-1}A\right)^* = A_{(\epsilon,p)}^{-1}A$, then $A_{(\epsilon,p)}^{-1} = A^+$;

else $A_{(\epsilon,p)}^{-1}$ is a right inverse of A;

b) For $n \leq m$, if $(AA_\epsilon^{-1})^* = AA_\epsilon^{-1}$, then $A_{(\epsilon,p)}^{-1} = A^+$,

else A_ϵ^{-1} is a left inverse of A .

Case 2. If $r_\epsilon(A) = rank(A) = r < \min\{m, n\}$ then:

Rule 2.1 If A satisfies condition (2), then $A_{(\epsilon,r)}^{-1}$ is the Moore-Penrose inverse of A.

Rule 2.2 If condition (2) does not hold, compute a full-rank factorization $A = PQ$ and select one of the following two rules.

Rule 2.3 If both P and Q satisfy condition (2), then $A_{(\epsilon,r)}^{-1} = A^+$;

Rule 2.4 If P or Q satisfies condition (2), then

a) $A_{(\epsilon,r)}^{-1}$ satisfies conditions (1.1), (1.2) and (1.3), if $m \leq n$;

b) $A_{(\epsilon,r)}^{-1}$ satisfies conditions (1.1), (1.2) and (1.4), if $m \geq n$.

Rule 2.5 If neither P nor Q satisfies (2), use Corollary 3.1.

Case 3. If $r_\epsilon(A) < rank(A)$ then $A_{(\epsilon,r)}^{-1} \notin A\{1,2\}$.

Example 4.2. Matrix $A = \begin{pmatrix} -1 & 1 & 2 \\ -1 & -4 & -3 \end{pmatrix}$ satisfies condition (2), so that

$$A_{(S,2)}^{-1} = A^+ = \begin{pmatrix} \frac{-7}{15} & \frac{-1}{5} \\ \frac{-2}{15} & \frac{-1}{5} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

Example 4.3. The rank-deficient matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ satisfies condition (2). According to rulle 2.1, $A_{(\epsilon,2)}^{-1}$ is the Moore-Penrose inverse of A

$$A^{(+,2)} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Example 4.4. Consider $A = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 14 & 11 \\ 2 & 10 & 8 \\ 0 & 2 & 2 \end{pmatrix}$. We have $rank(A) = 2$, and $det_2^S(A) = 54 \neq 0$. A full-rank factorization of A is $P = \begin{pmatrix} 1 & 3 \\ 3 & 11 \\ 2 & 8 \\ 0 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The matrix Q satisfies (2), so that $Q^{-1}_{(\epsilon,2)} = Q^{+}$. Also, $P^{-1}_{(\epsilon,2)} \neq P^{+}$, so that $A^{-1}_{(S,2)} = \begin{pmatrix} \frac{47}{54} & \frac{5}{18} & \frac{-14}{27} & \frac{-25}{27} \\ \frac{8}{27} & \frac{1}{9} & \frac{-4}{27} & \frac{-8}{27} \\ \frac{-31}{54} & \frac{-1}{6} & \frac{10}{27} & \frac{17}{27} \end{pmatrix}$ satisfies conditions (1.1), (1.2) and (1.4).

Example 4.5. Full-rank factorization of $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ is $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Using $P_\epsilon^{-1} \neq P^{+}$ and $Q_\epsilon^{-1} \neq Q^{+}$ it is easy to see that $A^{-1}_{(S,2)} = \begin{pmatrix} \frac{5}{6} & \frac{5}{12} & \frac{-1}{6} & \frac{-3}{4} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{12} & \frac{1}{6} & \frac{5}{12} \end{pmatrix} \in A\{1,2\}$.

Example 4.6. Consider matrices of the form $A = \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & -3 & -1 \end{pmatrix}$, If we use Stojaković's definition, it is easy to verify that $r_\epsilon(A) = 2 < rank(A)$. Moreover, $X = A^{-1}_{(S,2)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{8} & \frac{5}{4} & \frac{5}{8} \\ \frac{5}{8} & \frac{5}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{-1}{8} \end{pmatrix}$, and $AXA = \begin{pmatrix} 1 & \frac{-9}{8} & \frac{9}{8} & 3 \\ 0 & \frac{-7}{8} & \frac{7}{8} & 1 \\ 2 & \frac{31}{8} & \frac{-31}{8} & -1 \end{pmatrix} \neq A$, $\quad XAX = X$.

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Received by the editors May 10, 1994.