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SOME DESIGNS WITH PROJECTIVE SYMPLECTIC GROUPS AS AUTOMORPHISM GROUPS

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Abstract

Using a modification of the Kramer-Mesner method, 139 designs with pairwise distinct parameters and with a projective symplectic group as an automorphism group (i.e., as a subgroup of the full automorphism group) are constructed. Among them, there are 101 2-designs and two 3-designs over 15 points with $PSp(4,2)$ as an automorphism group and 36 2-designs over 40 points with $PSp(4,3)$ as an automorphism group. In particular, each of the two groups gives a Steiner system for $t = 2$. Multiple appearances of the constructed designs in orbit incidence matrices are counted.

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1. Introduction

An n-set is a set of cardinality n. Given a group G acting upon a ground-set, an *n-G-orbit* is an orbit of n-subsets of the ground-set, which arises by action of G. If G is fixed, then the denotation "n-G-orbit" will

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be shortened to "n-orbit". A t-(v,k,λ) *design* [4] is an incidence structure on the v-ground-set, which consists of some k-subsets (called *blocks*) of the ground-set, without repetitions, and which satisfies the property that each t-subset of the ground-set is contained in exactly λ blocks.

1.1. The Kramer-Mesner method

The well-known Kramer-Mesner method [9] for constructing t-(v,k,λ) designs with a prescribed automorphism group G works as follows:

Let λ_{ij} ([4], pp. 185) denote the number of elements of the j-th k-G-orbit, that contain an arbitrary fixed element of the i-th t-G-orbit, $t<k$. This notion is well-defined, since each t-set of a t-G-orbit is contained in the same number of k-sets of a k-G-orbit.

The matrix (λ_{ij}) will be denoted here as $\Lambda(G;t,k)$; the same matrix was denoted as $A(G;H;t,k)$ in [9] and as $A_{t,k}$ in [11]; this matrix can be called *orbit incidence matrix* for t-orbits and k-orbits by action of G. If $n(G,i)$ denotes the number of i-G-orbits, then the size of $\Lambda(G;t,k)$ is $n(G,t)\times n(G,k)$. The row sums in $\Lambda(G;t,k)$ are uniform and are equal to $\lambda_{max}=\begin{pmatrix} v-t \\ k-t \end{pmatrix}$.

The key idea of the method is to find a *proper* subset S (if exists) of columns of $\Lambda(G;t,k)$ with uniform row sums λ. Blocks of the required design are exactly all those k-subsets of the v-ground-set that belong to the k-G-orbits corresponding to columns of S. In other words, a t-(v,k,λ) design with automorphism group G can be recognized as a *proper* submatrix of $\Lambda(G;t,k)$ that consists of whole columns and has uniform row sums λ in all $n(G,t)$ rows. One easily concludes by using complementary submatrices that it suffices to search for $\lambda\le\frac{1}{2}\cdot\lambda_{max}$.

In this way, blocks of a t-(v,k,λ) design are obtained as k-sets belonging to a union of several k-G-orbits.

It is essential with the Kramer-Mesner method that it gives designs with a prescribed group as *an* automorphism group. This follows from the facts that the orbits are preserved under action of a group and that the family of blocks is composed of whole orbits. Note, however, that the prescribed group need not be the *full* automorphism group (see, e.g., [11]), that is, the group of *all* automorphisms of a design, the only one that may be denoted by Aut(design).

2. On the groups $PSp(2n,q)$

Let $V(2n,q)$ denote the *non-degenerate symplectic* $2n$–dimensional vector space over $GF(q)$, which is equipped with a bilinear alternative scalar product (x,y)

($(y,x)=-(x,y)$; there does not exist x s.t. $(x,y)=0$ for all $y\in V$).

It is known that $V(2n,q)$ can be represented as the orthogonal sum of n hyperbolic (2-dimensional) planes $H_1,...,H_n$. Each one of the hyperbolic planes has two vectors x, y with $(x,y)=1$, while each two vectors x, y from different planes satisfy that $(x,y)=0$.

Let $(x_1,...,x_{2n})$ and $(y_1,...,y_{2n})$ be the coordinates of vectors x and y w.r.t. a basis $(e_1,...,e_{2n})$ satisfying $(e_{2i-1},e_{2i})=1$ for $1\le i\le n$ and $(e_i,e_j)=0$ otherwise (the vectors e_{2i-1} and e_{2i} belong to H_i for $1\le i\le n$).

The bilinear alternative scalar product (x,y) is effectively constructed according to the following scheme:

$$\begin{aligned}(x,y)\;&=\sum_{i=1}^{n}[x_{2i-1}\cdot y_{2i}\cdot(e_{2i-1},e_{2i})+x_{2i}\cdot y_{2i-1}\cdot(e_{2i},e_{2i-1})]\\&=\sum_{i=1}^{n}(x_{2i-1}\cdot y_{2i}-x_{2i}\cdot y_{2i-1}).\end{aligned}$$

The $2n$-dimensional *symplectic group* over $GF(q)$, denoted by $Sp(2n,q)$, is the linear group which preserves bilinear alternative scalar product, that is, its elements are matrices M of the size $2n\times 2n$ over $GF(q)$, which satisfy $(x,y)=(xM,yM)$ for each $x,y\in V(2n,q)$.

Let the row-vectors of M be denoted by $r_1,...,r_{2n}$. The vectors xM and yM have the same coefficients w.r.t. the base $r_1,...,r_{2n}$ as x and y have w.r.t. the initial base $e_1,...,e_{2n}$. The preservance of scalar product implies that the vectors r_i should fulfil the same conditions w.r.t. the product as the vectors e_i do.

Thus the group $Sp(2n,q)$ consists of $2n\times 2n$ matrices over $GF(q)$

with row-vectors $r_1, ..., r_{2n}$, which satisfy $(r_{2i-1}, r_{2i}) = 1$ for $1 \leq i \leq n$ and $(r_i, r_j) = 0$ otherwise; the vectors r_{2i-1} and r_{2i} belong to the i-th hyperbolic plane, for $1 \leq i \leq n$.

It is easy to show ([8], Lemma 9.13. Chap. II) that the cardinality of $Sp(2n, q)$ is equal to

$$|Sp(2n, q)| = (q^{2n} - 1) \cdot q^{2n-1} \cdot (q^{2n-2} - 1) \cdot q^{2n-3} \cdot ... \cdot (q^2 - 1) \cdot q \ .$$

The group $PSp(2n, q)$ = $2n$-dimensional *projective symplectic group* over $GF(q)$ is the factor-group obtained from $Sp(2n, q)$ by reducing with the subgroup of homotethies $H = \{M \in Sp(2n, q), M = aE\}$.

Given $x, y \in V(2n, q)$ and $M = aE \in H$, it holds that $(xM, yM) = a^2(x, y)$. On the other hand, the definition of $Sp(2n, q)$ gives that $(xM, yM) = (x, y)$ and so $a^2 = 1$. This equation has only trivial solution with the fields of characteristic 2 and two solutions otherwise. Therefore the group H has only one element in the first case and two elements in the second one. Thus

$$|PSp(2n, q)| = \begin{cases} |Sp(2n, q)|, & q = 2^k \\ \frac{1}{2} \cdot |Sp(2n, q)|, & q \text{ odd prime power} \end{cases}$$

The group $PSp(2n, q)$ acts on the ground-set $P(2n-1, q)$ = $(2n-1)$-dimensional projective space over $GF(q)$. The elements of $P(2n-1, q)$ (projective points) correspond to the 1-dimensional subspaces of $V(2n, q)$.

The projective space $P(2n-1, q)$ can be represented by the vectors from $V(2n, q)$ of the form $(a_1, ..., a_{2n-k-1}, 1, \underbrace{0, ..., 0}_{k})$, for $k = 0, ..., 2n-1$; these vectors are canonical representatives of homotethy classes = projective points. It is obvious that $|P(2n-1, q)| = \dfrac{q^{2n} - 1}{q - 1}$; non-zero vectors of $V(2n, q)$ are partitioned into this number of homotethy classes.

The algorithm for generating matrices of $PSp(2n, q)$ has the following scheme:

BEGIN of algorithm

Choose r_1 to be an arbitrary element of $P(2n-1,q)$, that is, the canonical representative of a homotethy class within $V(2n,q)$;

(generally speaking, such a choice of the first row of matrices is sufficient to establish projectivity of matrices of any projective linear group)

Choose r_2 to be an arbitrary element of $V(2n,q)$ such that $(r_1,r_2)=1$;

(thus the projectivity is taken into account and the representatives of the first hyperbolic plane are chosen)

FOR $i:=2$ TO n DO BEGIN

Choose r_{2i-1} to be an arbitrary non-zero element of $V(2n,q)$ such that
$(r_j,r_{2i-1})=0$ for $1\le j\le 2i-2$;

(the first representative of the i-th hyperbolic plane is chosen)

Choose r_{2i} to be an arbitrary non-zero element of $V(2n,q)$ such that
$(r_{2i-1},r_{2i})=1$ and $(r_j,r_{2i})=0$ for $1\le j\le 2i-2$;

(the second representative of the i-th hyperbolic plane is chosen)

END
END of algorithm

We shall consider only the case $n=2$. The points of $P(3,q)$ will be throughout denoted by natural numbers $1,2,...,q^3+q^2+q+1$. More precisely, the following values $2n$ and q will be considered in this paper:

$2n$	q	$\|P(2n-1,q)\|$	$\|PSp(2n,q)\|$
4	2	15	720
4	3	40	25920

The groups $PSp(4,2)$ and $PSp(4,3)$ act on the 3-dimensional projective spaces of order 2 (and size 15), respectively of order 3 (and size 40).

The group $PSp(4,2)$ coincides with the group $Sp(4,2)$. The special construction of the row-vector r_1 coincides with a general construction in that case, since each homotethy class within $V(4,2)$ contains a single vector. On the other hand, there are two vectors in each homotethy class of $V(4,3)$ and two matrices in each homotethy class of $Sp(4,3)$. Thus the reduction factor in transition from $Sp(4,3)$ to $PSp(4,3)$ (the factor obtained by introducing projectivity into $Sp(4,3)$) is equal to 2. This reduction factor remains valid with all groups $SP(2,q)$, where q is an odd prime power. Although the cardinality of a homotethy class (1-dimensional subspace) within a space $V(2n,q)$ is equal to $q-1$, the cardinality of a homotethy class within (matrices of) $Sp(2,q)$ remains equal to 2 for q odd prime power and equal to 1 for q of the form 2^k.

Further, the group $PSp(4,2)$ is known to be isomorphic with the symmetric group S_6 ([8], Chap II. 9.21, pp. 227). Nevertheless, its action is not trivial, since it acts on the ground-set of cardinality 15.

A modification of the Kramer-Mesner method has been applied to the projective symplectic groups $G = PSp(4,2)$ with $2 \le t < k \le 6$ and $G = PSp(4,3)$ with $2 \le t < k \le 4$. Such a choice (and restriction) of groups and design parameters to be considered was motivated by the following arguments:

- The group $PSp(2,q)$ is isomorphic to the projective special linear group $PSL(2,q)$ (a consequence of Lemma 9.12, Chap. II, pp. 219, [8]). Designs with $PSL(2,q)$ as an automorphism group are much more investigated (e.g., [12]).
- Groups $PSp(4,4)$ and $PSp(6,2)$ are too large; according to the above formula, their cardinalities are 979.200 and 1.451.520, respectively.
- Number of 7-$PSp(4,2)$-orbits and 5-$PSp(4,3)$-orbits is too high to allow an exhaustive search for designs in a reasonable time.

These groups have been used for a computer-aided searching t-$(15,k,\lambda)$ designs, respectively t-$(40,k,\lambda)$ designs.

2.1. Reduced orbits

Given an h-homogeneous group G ($h < t < k$), the matrix $\Lambda(G;t,k)$ can be computed by using only those t-subsets and k-subsets of the ground-set, which contain a fixed h-subset FS of the ground-set.

Namely, when constructing k-G-orbits, it suffices to consider only those $\binom{q+1-h}{k-h}$ k-subsets of the ground-set, which are supersets of the h-

subset FS; we call these k-subsets "special". "Special" k-subsets are proportionally distributed among k-G-orbits; the number of "special" k-subsets within a k-G-orbit is obtained by multiplying the total number of its k-subsets by $\binom{q+1-h}{k-h} / \binom{q+1}{k}$; the result of this multiplication must be an integer. "Special" k-subsets within a k-G-orbit constitute a *reduced* k-G-orbit. An analogous reduction is applied to t-G-orbits.

Reduced k-G-orbits are constructed by applying elements of G to their representative k-subsets; the image k-subsets are recorded iff they are special. It suffices to keep in computer memory the ordinal numbers of k-G-orbits containing "special" k-subsets, together with the ordinal numbers of these k-subsets in the lexicographic order.

Reduced t-G-orbits and reduced k-G-orbits are sufficient for construction of the matrix $\Lambda(G;t,k)$, since set-inclusion preserves "speciality"; that is, all k-supersets of a "special" t-subset are "special" k-subsets.

Groups $PSp(2n,q)$ operate transitively (hence also 1-homogeneously) on the ground-set $P(2n-1,q)$ [8] (Theorem 9.15 c), pp. 221.). We shall choose $FS = \{1\}$ with projective symplectic groups.

3. Designs

The main result of this paper reads:

Theorem 1. *There exist*

a) *t-(15,k,λ) designs with* $PSp(4,2)$ *as an automorphism group and with* $(t,k) \in \{(2,3),(2,4),(2,5),(2,6),(3,5),(3,6)\}$.

b) *t-(40,k,λ) designs with* $PSp(4,3)$ *as an automorphism group and with* $(t,k) \in \{(2,3),(2,4)\}$.

There are 103 distinct values of λ *in case* **a)** *and 36 distinct values of* λ *in case*

b) *(these values are given below).*

Proof. The proof will be given by exhibiting:

– data necessary to identify the orbits under action of the considered groups;

  (Tables 1-8, subsection 3.1.)

– matrices $\Lambda(PSp(4,2);t,k)$ for $2 \le t < k \le 6$ and $\Lambda(PSp(4,3);t,k)$ for $2 \le t < k \le 4$; (Tables 9-16, subsection 3.2.)

Column combinations (sets of columns) of orbit incidence matrices, which correspond to the designs, will be explicitly given only in those cases that we find to be particularly interesting.

3.1. Orbits

It turns out that there are 2 2-orbits, 5 3-orbits, 9 4-orbits, 15 5-orbits, and 21 6-orbits under action of $PSp(4,2)$. In addition, there are 2 2-orbits, 5 3-orbits, and 16 4-orbits under action of $PSp(4,3)$.

In accordance with the discussion in subsection 2.1., 1-homogenicity of the groups $PSp(4,q)$ enables the representatives of all orbits to be "special" supersets of a fixed 1-set $FS = \{1\}$.

In order to enable identification of the orbits of 4-subsets and 6-subsets by action of the groups, associated with rows and columns of the matrices, the following data will be given in Tables 1-8 for these orbits:

– the ordinal number of an orbit, which is associated to the corresponding row (column) of a matrix $\Lambda(PSp(4,q);t,k)$.

– the elements of the lexicographically the first "special" representative, apart from the compulsory element 1.

– the number of "special" subsets within an orbit.

Example. The denotations | 2 | 2 9 | 4 | in Table 2 and | 10 | 2 3 5 14 | 120 | in Table 4 mean that the 2nd 3-orbit of $PSp(4,2)$ contains the representative $\{1,2,9\}$ and the total of 4 "special" 3-subsets, while the 10th 5-orbit of $PSp(4,2)$ contains the representative $\{1,2,3,5,14\}$ and the total of 120 "special" 5-subsets.

1	2	8	2	3	6

Table 1. Data for 2-orbits of $PSp(4,2)$

1	2	3	36	2	2	9	4	3	2	10	12
4	2	13	36	5	3	13	3				

Table 2. Data for 3-orbits of $PSp(4,2)$

1	2	3	4	12	2	2	3	5	96	3	2	3	6	96
4	2	3	9	48	5	2	3	13	48	6	2	9	13	16
7	2	10	11	8	8	2	10	13	16	9	2	13	14	24

Table 3. Data for 4-orbits of $PSp(4,2)$

1	2	3	4	5	120	2	2	3	4	9	30	3	2	3	4	13	15
4	2	3	5	7	40	5	2	3	5	8	120	6	2	3	5	9	60
7	2	3	5	10	120	8	2	3	5	12	30	9	2	3	5	13	120
10	2	3	5	14	120	11	2	3	6	13	120	12	2	3	6	14	24
13	2	3	9	13	60	14	2	9	13	14	20	15	2	10	11	12	2

Table 4. Data for 5-orbits of $PSp(4,2)$

1	2	3	4	5	6	24	2	2	3	4	5	7	72	3	2	3	4	5	8	72
4	2	3	4	5	9	144	5	2	3	4	5	10	144	6	2	3	4	5	11	144
7	2	3	4	5	12	144	8	2	3	4	5	13	144	9	2	3	4	9	10	6
10	2	3	4	9	13	36	11	2	3	5	7	9	24	12	2	3	5	7	10	144
13	2	3	5	7	13	144	14	2	3	5	8	10	288	15	2	3	5	8	12	72
16	2	3	5	8	13	144	17	2	3	5	8	14	144	18	2	3	5	9	14	36
19	2	3	5	10	15	48	20	2	3	6	13	15	24	21	2	9	13	14	15	4

Table 5. Data for 6-orbits of $PSp(4,2)$

2	1	27	4	2	12

Table 6. Data for 2-orbits of $PSp(4,3)$

1	2	3	27	2	2	4	324	3	2	6	216
4	2	37	162	5	4	7	12				

Table 7. Data for 3-orbits of $PSp(4,3)$

1	2	3	4	864	2	2	3	28	9	3	2	3	29	288	4	2	3	37	144
5	2	4	5	216	6	2	4	6	432	7	2	4	7	144	8	2	4	10	2592
9	2	4	11	324	10	2	4	20	1296	11	2	4	21	1296	12	2	4	37	432
13	2	6	12	648	14	2	6	37	288	15	2	37	38	162	16	4	7	37	4

Table 8. Data for 4-orbits of $PSp(4,3)$

3.2 Orbit incidence matrices

Results of our search are the following:

The group $PSp(4,2)$ gives:

- a 2-(15,3,1) design;
- 2-(15,4,λ) designs for $\lambda \in \{6,24,30,36\}$;
- 2-(15,5,λ) designs for 49 distinct values of $\lambda \leq \lambda_{max}/2 = 286/2$;
- 2-(15,6,λ) designs for 47 distinct values of $\lambda \leq \lambda_{max}/2 = 715/2$;
- a 3-(15,5,30) design;
- a 3-(15,6,100) design.

The group $PSp(4,3)$ gives:

- a 2-(40,3,2) design and a 2-(40,3,18) design;

- 2-(40,4,λ) designs for 34 distinct values of $\lambda \leq \lambda_{max}/2 = 703/2$.

This subsection contains the used orbit incidence matrices. These matrices will be in some cases listed together with some of the corresponding designs. The designs are denoted with 0-1 incidence vectors for columns. These vectors are written below the matrices and are separated from them by horizontal lines. The blocks of a design are exactly those k-subsets of the ground-set, that belong to the union of k-orbits corresponding to the columns denoted by 1 in the incidence vectors. For example, blocks of the only $2\text{-}(15,3,1)$ design in Table 9 are exactly all those 3-subsets of the ground-set $\{1,2,...,15\}$ that belong to the union of the 2nd and the 5th 3-orbit.

In addition to the matrix $\Lambda(PSp(4,q);t,k)$, the following data are given under the title "LISTS", for each one of the considered pairs (t,k):

- the parameters t, $v = q^3+q^2+q+1$ and k of the constructed designs;
- the value of $\lambda_{max} = \binom{q^3+q^2+q+1-t}{k-t}$;
- the total number of column combinations in the orbit incidence matrix, which correspond to the designs (they are shortly denoted as "designs" below);
- the number of corresponding distinct λ-values $\leq \lambda_{max}/2$;
- ordered pairs $(\lambda, frequency(\lambda))$, where $frequency(\lambda)$ denotes the number of column combinations corresponding to $t\text{-}(q^3+q^2+q+1,k,\lambda)$ designs for a fixed λ. For example, data (6,2) (24,2) (30,2) (36,3) in Table 10 mean that the number of column combinations of $\Lambda(PSp(4,2);2,4)$ corresponding to 2-(15,4,λ) designs is equal to 2, 2, 2, 3, for λ equal to $6,24,30,36$ respectively.

We point out the necessity to distinguish three meanings of the word "design", which correspond to three levels of a hierarchy:

parameter level – designs determined up to the parameters

This meaning is most generally known. The existence question for designs corresponding to the quadruples $t\text{-}(v,k,\lambda)$ is by far the most interesting one.

isomorphism level – designs determined up to an isomorphism

This meaning is related to the design enumeration problem, which has been solved with a very small number of the known design parameters.

column combination level –

This meaning is a speciality of the Kramer-Mesner method. Each design obtained with this method corresponds to a column combination with uniform row sums λ within the orbit incidence matrix.

Parameters of a design that corresponds to a column combination are immediately known. However, the isomophism question for some two designs with the same parameters, that correspond to some two distinct column combinations, remains hard.

The denotations "design(s)" in the lists below, as well as the notion of "frequency of designs with some λ-value", are used in the third, specific, meaning.

LISTS

2-(15,3,λ): $\lambda_{max} = 13$; 1 design.

6	1	3	3	0
4	0	0	8	1
0	1	0	0	1

Table 9. Matrix $\Lambda(PSp(4,2);2,3)$ and a $2\text{-}(15,3,1)$ design

Frequency: (1,1) .

2-(15,4,λ): $\lambda_{max} = 78$; 9 designs with 4 distinct λ-values $\leq \lambda_{max}/2$.

3	24	18	15	6	3	3	3	3
2	16	24	4	16	4	0	4	8

Table 10. Matrix $\Lambda(PSp(4,2);2,4)$

Frequencies: (6,2) (24,2) (30,2) (36,3) .

2- (15,5,λ): λ_{max} = 286; 310 designs with 49 distinct λ-values $\leq$ $\lambda_{max}/2$.

36	12	3	14	30	24	42	9	30	36	24	6	15	4	1
32	4	6	8	40	8	24	8	40	32	48	8	20	8	0

Table 11. Matrix $\Lambda(PSp(4,2);2,5)$

Frequencies:

(16,1) (22,1) (24,1) (26,1) (30,2) (32,1) (34,1) (36,1) (40,3)
(42,1) (44,1) (46,3) (52,4) (54,3) (56,2) (60,5) (62,2) (64,2)
(66,2) (70,4) (72,3) (74,4) (76,7) (80,7) (82,10) (84,10) (86,5)
(90,11) (92,8) (94,5) (96,6) (100,6) (102,4) (104,4) (106,13)
(110,8) (112,16) (114,16) (116,8) (120,18) (122,10) (124,10)
(126,10) (130,9) (132,10) (134,8) (136,18) (140,9) (142,16) .

2- (15,6,λ): λ_{max} = 715; 12674 designs with 47 distinct λ-values $\leq \lambda_{max}/2$.

9	27	24	66	60	54	48	42	3	12	12	60	54	96	24	42	42	15	18	6	1
8	24	28	32	40	48	56	64	1	14	4	40	48	112	28	64	64	10	16	12	2

Table 12. Matrix $\Lambda(PSp(4,2);2,6)$

Frequencies:

(10,1) (15,1) (25,2) (30,3) (40,6) (45,6) (55,10) (60,13) (70,15) (75,15) (85,16)
(90,20) (100,26) (105,28) (115,36) (120,40) (130,56) (135,52) (145,74)
(150,108) (160,110) (165,150) (175,146) (180,180) (190,240) (195,210)
(205,282) (210,264) (220,307) (225,306) (235,352) (240,455) (250,417)
(255,512) (265,462) (270,503) (280,630) (285,543) (295,675) (300,592)
(310,620) (315,624) (325,646) (330,785) (340,667) (345,800) (355,668)
.

3-(15,5,λ): $\lambda_{max} = 66$; 1 design.

12	3	1	2	6	4	10	2	8	8	6	2	2	0	0
0	9	0	0	0	9	18	0	0	18	0	0	9	3	0
6	3	0	8	6	12	12	3	6	6	0	0	3	0	1
6	0	1	2	12	1	4	2	8	8	12	2	5	3	0
0	0	6	0	0	0	0	0	24	0	24	0	12	0	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
0	1	1	0	1	0	1	1	1	0	0	0	0	1	1

Table 13. Matrix $\Lambda(PSp(4,2);3,5)$ and a 3-(15, 5, 30) design
Frequency: (30,1) .

3-(15,6,λ): $\lambda_{max} = 220$; 1 design.

4	8	8	20	20	20	18	16	1	3	2	16	14	28	6	14	10	4	6	2	0
0	0	0	36	36	18	0	0	3	9	3	18	18	36	9	0	18	9	6	0	1
2	12	6	30	18	12	12	6	1	3	10	30	24	24	6	6	6	6	6	0	0
2	8	10	6	10	14	16	18	0	4	1	12	14	36	11	22	24	3	4	4	1
0	0	0	0	0	0	24	48	0	12	0	0	24	48	0	24	24	0	8	8	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
0	1	1	1	0	0	1	0	1	1	0	0	0	1	1	0	0	0	1	1	1

Table 14. Matrix $\Lambda(PSp(4,2);3,6)$ and a 3-(15, 6, 100) design
Frequency: (100,1) .

2- (40,3,λ): $\lambda_{max} = 38$; 2 designs with 2 distinct λ-values $\leq \lambda_{max}/2$.

2	16	16	4	0
0	18	0	18	2
---	---	---	---	---
1	0	0	0	1
1	1	0	0	0

Table 15. Matrix $\Lambda(PSp(4,3);2,3)$
a 2-(40, 3, 2) design and a 2-(40, 3, 18) design

Frequencies: (2,1) (18,1) .

2- (40,4,λ): $\lambda_{max} = 703$; 156 designs with 34 distinct λ-values $\leq \lambda_{max}/2$.

80	1	32	8	16	40	8	192	24	72	120	16	72	16	6	0
36	0	0	18	18	18	18	216	27	162	54	72	0	36	27	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 16. Matrix $\Lambda(PSp(4,3);2,4)$ and a 2-$(40,4,1)$ design

Frequencies:

(1,1) (46,3) (63,3) (72,4) (73,4) (81,4) (118,4) (126,4) (127,4) (135,1) (136,3) (144,6) (145,6) (153,1) (190,6) (198,2) (199,2) (207,7) (208,3) (216,4) (217,4) (225,7) (262,3) (270,6) (271,6) (279,6) (280,4) (288,9) (289,9) (297,1) (334,13) (342,3) (343,3) (351,10) .

We have checked that the matrices $\Lambda(PSp(4,2);3,4)$, $\Lambda(PSp(4,2);4,5)$, $\Lambda(PSp(4,2);4,6)$, and $\Lambda(PSp(4,3);3,4)$, have no proper submatrices with uniform row sums. Consequently, we cannot use the Kramer-Mesner method for searching 3-$(15,4,\lambda)$, 4-$(15,5,\lambda)$, 4-$(15,6,\lambda)$ and 5-$(15,6,\lambda)$ designs with $PSp(4,2)$ as an automorphism group, as well as for searching 3-$(40,4,\lambda)$ designs with $PSp(4,3)$ as an automorphism group.

4. Data on some former constructions

Results of this paper show that the designs constructed here have $PSp(4,2)$, respectively $PSp(4,3)$, as an automorphism group. An early draft of this paper has been presented in [14]. We have applied some other modifications of the Kramer-Mesner method in the papers [1], [2] and [3].

In this section we point out some data, taken from [6] and [4], on other constructions of some of the designs with the same parameters as those that we have found by using groups $PSp(4,2)$ and $PSp(4,3)$. It turned out that the existence of designs over 15 points with the obtained parameters, as well as the existence of a Steiner system 2-(40,4,1), has already been known.

The cited data are related to somewhat wider classes of design parameters. For example, it has been formerly found (a consequence of the results from [5]) that there are 2-(15,6,5s) designs with 71 distinct values of s ($1 \leq s \leq 71$). However, our construction gives that only 47 of these designs have $PSp(4,2)$ as an automorphism group.

On the other hand, we have constructed two 2-$(40,3,\lambda)$ designs, and 33 2-$(40,4,\lambda)$ designs with pairwise distinct parameters and with $\lambda > 1$. We are lacking information on former constructions of designs with these parameters.

Among the constructed designs, there are two Steiner systems:

2-(15,3,1) and 2-(40,4,1).

According to [4] (Table F. "Series of Steiner systems", pp. 640, mainly due to [16]), these two Steiner systems belong to two infinite families of Steiner systems:
2-(v,3,1) for $v =$ 1 or 3 mod 6, and 2-(v,4,1) for $v =$ 1 or 4 mod 12. Moreover, a complete enumeration of 80 non-isomorphic Steiner systems 2-(15,3,1) has been completed in [13].

The denotation "$X \longleftarrow Y$" in Table 17. means that the existence of a design X follows from the existence of a design Y. The denotation "$X \Longleftarrow Y$" in the same table means that the existence of a design X has been proved in the reference Y.

For $1 \leq s \leq 6$, 2-(15,3,s) $\Longleftarrow$ [7].

For $1 \leq s \leq 6$,

2-(15,4,6s) $\longleftarrow$ 3-(16,5,6s) $\longleftarrow$ 4-(17,6,6s) $\longleftarrow$ 5-(18,7,6s) $\Longleftarrow$ [10].

For $2 \leq s \leq 71$, 2-(15,5,2s) $\longleftarrow$ 3-(16,6,2s).

For $2 \leq s \leq 5$, 3-(16,6,2s) $\Longleftarrow$ [5].

For $6 \leq s \leq 71$, 3-(16,6,2s) $\longleftarrow$ 4-(17,7,2s) $\Longleftarrow$ [5].

For $1 \leq s \leq 71$, 2-(15,6,5s) $\Longleftarrow$ [5].

For $2 \leq s \leq 5$, 3-(15,5,6s), $\longleftarrow$ 4-(16,6,6s).

4-(16,6,30) $\Longleftarrow$ [5].

For $2 \leq s \leq 4$, 4-(16,6,6s) $\longleftarrow$ 5-(17,7,6s).

For $s \in \{2,4\}$, 5-(17,7,6s) $\Longleftarrow$ [5].

5-(17,7,18) $\Longleftarrow$ [15].

For $1 \leq s \leq 5$, 3-(15,6,20s) $\longleftarrow$ 4-(16,7,20s).

For $s = 1,2,3,5$, 4-(16,7,20s) $\Longleftarrow$ [5].

4-(16,7,80) $\longleftarrow$ 5-(17,8,80) $\Longleftarrow$ [10].

Table 17. Origin of some design parameters

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