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SUBMANIFOLDS OF THE MANIFOLD WITH AN $f(3,\varepsilon)$-STRUCTURE

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Abstract

We consider an $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ having L and M as complementary distributions and study its invariant submanifolds.

Two cases of invariant submanifolds are considered. In the first case we get an induced almost complex structure or almost product structure, and in the second case we obtain an induced $\tilde{f}(3,\varepsilon)$-structure on the invariant submanifold. It is shown that the complementary distribution of induced $\tilde{f}(3,\varepsilon)$-structure in the submanifold embedded as an invariant submanifold of $\mathcal{M}^n$ and the induced $\tilde{f}(3,\varepsilon)$-structure are all integrable if the corresponding distributions and the structure are integrable in $\mathcal{M}^n$.

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1. Introduction

Let $\mathcal{N}^p$ be a p-dimensional C^∞ manifold embedded differentially as a submanifold in an n-dimensional C^∞-Riemannian manifold $\mathcal{M}^n$. Let ϕ be an embedding map $\phi : \mathcal{N}^p \to \mathcal{M}^n$, and $\phi_*(\equiv B)$ the Jacobian map of ϕ i.e. $B : T(\mathcal{N}^p) \to T(\mathcal{M}^n)$. Denote by $T(\mathcal{N},\mathcal{M})$ the set of all vectors tangent to the submanifold $\phi(\mathcal{N}^p)$. It is known that $B : T(\mathcal{N}^p) \to T(\mathcal{N},\mathcal{M})$ is an isomorphism [2].

Take the C^∞ vector fields $\tilde{X}$ and $\tilde{Y}$ which are tangential to $\phi(\mathcal{N}^p)$. Let X and Y be the local C^∞ extension of X and Y respectively.

The restriction of $[X,Y]$ to $\phi(\mathcal{N}^p)$, i.e. $[X,Y]|_{\phi(\mathcal{N}_p)}$ is determined independently of the choice of these local extensions X and Y. We can write

$$[\tilde{X},\tilde{Y}] = [X,Y]|_{\phi(\mathcal{N}^p)}.$$

Since B is an isomorphism, we have

$$[B\tilde{X},B\tilde{Y}] = B[\tilde{X},\tilde{Y}] \text{ for all } \tilde{X},\tilde{Y} \in T(\mathcal{N}^p).$$

We define the induced metric $\tilde{g}$ on $\mathcal{N}^m$ as follows

$$\tilde{g}(\tilde{X},\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) \text{ for all } \tilde{X},\tilde{Y} \in T(\mathcal{N}^p), \tag{1.1}$$

where g is the Riemannian metric in $\mathcal{M}^n$. It can be easily verified that $\tilde{g}$ is the Riemanian metric in $\mathcal{N}^p$.

We assume that $\mathcal{M}^n$ is the manifold with the structure tensor $f(3,\varepsilon)$ of the rank $r \leq n$. $\mathbf{l} = \varepsilon f^2$, and $\mathbf{m} = I - \varepsilon f^2$, $(\varepsilon = \pm 1)$ are complementary projection operators corresponding to which L and M are complementary distributions of dimension r and $n-r$ respectively.

As in [3], we have

$$\begin{aligned} &f^3 = \varepsilon f,\ (\varepsilon = \pm 1),\ f\mathbf{l} = \mathbf{l}f = f,\ f\mathbf{m} = \mathbf{m}f = 0,\\ &f^2\mathbf{l} = \mathbf{l}f^2 = \varepsilon \mathbf{l},\ f^2\mathbf{m} = \mathbf{m}f^2 = 0. \end{aligned} \tag{1.2}$$

2. Invariant submanifolds of $f(3,\varepsilon)$-structure manifold

Definition 2.1. *$\mathcal{N}^p$ is said to be an invariant submanifold of $\mathcal{M}^n$ if the tangent space $T_u(\phi(\mathcal{N}^p))$ of $\phi(\mathcal{N}^p)$ is invariant by the linear mapping f at each point u of $\phi(\mathcal{N}^p)$ so that for each $\tilde{X} \in T(\mathcal{N}^p)$ we have $f(B\tilde{X}) = B\tilde{X}'$, for some $\tilde{X}' \in T(\mathcal{N}^p)$.*

If we define a $(1,1)$ tensor field $\tilde{f}$ in $\mathcal{N}^p$ by $\tilde{f}(\tilde{X}) = \tilde{X}'$, as in [4], then we have

$$f(B\tilde{X}) = B(\tilde{f}\tilde{X}). \tag{2.1}$$

In the first case we assume that the distribution M is never tangential to $\phi(\mathcal{N}^p)$ i. e. no vector field of type $\mathbf{m}X$, $X \in T(\phi(\mathcal{N}^p))$ is tangential to $\phi(\mathcal{N}^p)$. It shows that any vector field of type $\mathbf{m}X$ is independent of any vector field of the form $B\tilde{X}$, $\tilde{X} \in T(\mathcal{N}^p)$.

Applying f to (2.1) we have

$$f^2(B\tilde{X}) = B(\tilde{f}^2\tilde{X}). \tag{2.2}$$

Now we shall show that vector fields of the type $B\tilde{X}$ belong to the distribution L in this case. If we suppose that $\mathbf{m}(B\tilde{X}) \neq 0$, then $\mathbf{m}(B\tilde{X}) = (I_{T(\mathcal{M}^n)} - \varepsilon f^2)(B\tilde{X}) = B\tilde{X} - \varepsilon f^2(B\tilde{X}) = B(\tilde{X} - \varepsilon\tilde{f}^2\tilde{X})$, which, contrary to our assumption, shows that $\mathbf{m}(B\tilde{X})$ is tangential to $\phi(\mathcal{N}^p)$. Therefore $\mathbf{m}(B\tilde{X}) = 0$.

Now, since $\mathbf{l} = \varepsilon f^2$, from (2.2) and (1.2) we have $B(\tilde{f}^2\tilde{X}) = f^2(B\tilde{X}) = \varepsilon\mathbf{l}(B\tilde{X}) = \varepsilon(I_{T\mathcal{M}^n)} - \mathbf{m})(B\tilde{X}) = \varepsilon B\tilde{X} - \varepsilon\mathbf{m}(B\tilde{X})$, $B(\tilde{f}^2\tilde{X}) = B(\varepsilon\tilde{X})$, which in view of B being an isomorphism, gives $\tilde{f}^2(\tilde{X}) = \varepsilon\tilde{X}$.

The tensor field $\tilde{f}$ in $\mathcal{N}^p$ defines an induced almost complex structure, or almost product structure, for $\varepsilon = -1$ or $+1$.

Theorem 2.1. *An invariant submanifold $\mathcal{N}^p$ in an $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ such that distribution M is never tangential to $\phi(\mathcal{N}^p)$ is an almost complex manifold or almost product manifold according as $\varepsilon = -1$ or $+1$.*

Let g be a Riemannian metric on $\mathcal{M}^n$ satisfying

$$g(X,Y) = g(fX, fY) + g(\mathbf{m}X, Y). \tag{2.3}$$

Theorem 2.2. *Let $\mathcal{N}^p$ be an invariant submanifold embedded in an $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ such that the distribution M is never tangential to $\phi(\mathcal{N}^p)$. If g denotes a Riemannian metric on $\mathcal{M}^n$ given by (2.3), then the induced metric $\tilde{g}$ and $\mathcal{N}^p$ defined by (1.1) is Hermitian.*

Proof.

$$\tilde{g}(\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}) = g(B\tilde{f}\tilde{X}, B\tilde{f}\tilde{Y}) = g(fB\tilde{X}, fB\tilde{Y}) =$$
$$= g(B\tilde{X}, B\tilde{Y}) - g(\mathbf{m}B\tilde{X}, B\tilde{Y}) = g(B\tilde{X}, B\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

In the second case we assume that the distribution M is always tangential to $\phi(\mathcal{N}^p)$. It follows, therefore, that $\mathbf{m}(B\tilde{X}) = B\tilde{X}^0$, where $\tilde{X} \in T(\mathcal{N}^p)$ for some $\tilde{X}^0 \in T(\mathcal{N}^p)$.

Let us define a $(1,1)$ tensor field $\tilde{\mathbf{m}}$ in $\mathcal{N}^p$ such that $\tilde{\mathbf{m}}\tilde{X} = \tilde{X}^0$.

We can write $\mathbf{m}(B\tilde{X}) = B(\tilde{\mathbf{m}}\tilde{X})$. Define a $(1,1)$ tensor field $\tilde{\mathbf{l}}$ in $\mathcal{N}^p$ by $\tilde{\mathbf{l}} = \varepsilon\tilde{f}^2$.

Then $(B\tilde{\mathbf{l}}\tilde{X}) = B(\varepsilon\tilde{f}^2\tilde{X}) = \varepsilon B(\tilde{f}^2\tilde{X}) = \varepsilon f^2(B\tilde{X}) = \mathbf{l}(B\tilde{X})$. Thus we have $B(\tilde{\mathbf{l}}\tilde{X}) = \mathbf{l}(B\tilde{X})$.

Theorem 2.3. *The* $(1,1)$ *tensor fields* $\tilde{\mathbf{l}}$ *and* $\tilde{\mathbf{m}}$ *in* $\mathcal{N}^p$ *satisfy the following*

$$\tilde{\mathbf{l}} + \tilde{\mathbf{m}} = I_{(T(\mathcal{N}^p))},\ \tilde{\mathbf{l}}\tilde{\mathbf{m}} = \tilde{\mathbf{m}}\tilde{\mathbf{l}} = 0,\ \tilde{\mathbf{l}}^2 = \tilde{\mathbf{l}},\ \tilde{\mathbf{m}}^2 = \tilde{\mathbf{m}}. \tag{2.4}$$

Proof. We have $\mathbf{l} + \mathbf{m} = I_{(T(\mathcal{M}^p))}$ i.e. $(\mathbf{l}+\mathbf{m})(B\tilde{X}) = B\tilde{X}$. Thus we have

$$B(\tilde{\mathbf{l}}\tilde{X}) + B(\tilde{\mathbf{m}}\tilde{X}) = B\tilde{X},\ B(\tilde{\mathbf{l}} + \tilde{\mathbf{m}})(\tilde{X}) = B\tilde{X}.$$

Therefore $\tilde{\mathbf{l}} + \tilde{\mathbf{m}} = I$, in view of the fact that B is an isomorphism.

Similary, we can prove other relations.

The relations (2.4) show that $\tilde{\mathbf{l}}$ and $\tilde{\mathbf{m}}$ are complementary projection operators in $\mathcal{N}^p$ given by $\tilde{\mathbf{l}} = \varepsilon\tilde{f}^2$ and $\tilde{\mathbf{m}} = I - \varepsilon\tilde{f}^2$. Moreover, from (2.1) we get $B(\tilde{f}^3\tilde{X}) = f^3(B\tilde{X}) = \varepsilon f(B\tilde{X}) = \varepsilon B(\tilde{f}\tilde{X})$. Thus we have $\tilde{f}^3 = \varepsilon\tilde{f}$, which shows that in this case $\tilde{f}$ defines an $\tilde{f}(3,\varepsilon)$-structure on $\mathcal{N}^p$ which we call induced $\tilde{f}(3,\varepsilon)$-structure. Further, from (1.1), (2.1), and (2.3) we have

$$\begin{aligned}\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + \tilde{g}(\mathbf{m}\tilde{X},\tilde{Y}) &= g(B(\tilde{f}\tilde{X}),B(\tilde{f}\tilde{Y})) + g(B\tilde{\mathbf{m}}\tilde{X},B\tilde{Y}) = \\ &= g(f(B\tilde{X}),f(B\tilde{Y})) + g(\tilde{\mathbf{m}}B\tilde{X},B\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}).\end{aligned}$$

i. e.

$$\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + \tilde{g}(\tilde{\mathbf{m}}\tilde{X},\tilde{Y}). \tag{2.5}$$

Theorem 2.4. *In invariant submanifold* $\mathcal{N}^p$ *embedded in an* $f(3,\varepsilon)$*-structure manifold* $\mathcal{M}^n$ *such that the distribution* M *is always tangential to* $\phi(\mathcal{N}^p)$ *there exists an induced* $\tilde{f}(3,\varepsilon)$*-structure manifold which admits a similar Riemanian metric* $\tilde{g}$ *satisfying (2.5)*

3. Integrability conditions

Theorem 3.1. *The Nijenhuis tensors N and $\tilde{N}$ of $\mathcal{M}^n$ and $\mathcal{N}^p$ respectively are related, as in [1], in the following way*

$$N(B\tilde{Y},B\tilde{Y}) = B\tilde{N}(\tilde{X},\tilde{Y}).$$

Proof. We have

$$\begin{aligned} N(B\tilde{X},B\tilde{Y}) &= [f(B\tilde{X})f(B\tilde{Y})] - f[B\tilde{X},f(B\tilde{Y})] - f[f(B\tilde{X}),B\tilde{Y}] + \\ &+ f^2[B\tilde{X},B\tilde{Y}] = \\ &= B[\tilde{f}\tilde{X},\tilde{f}\tilde{Y}] - B\tilde{f}[\tilde{X},\tilde{f}\tilde{Y}] - B\tilde{f}[\tilde{f}\tilde{X},\tilde{Y}] + B\tilde{f}^2[X,Y] = \\ &= B\tilde{N}(\tilde{X},\tilde{Y}). \end{aligned}$$

We can easily verify the following relations

$$\begin{aligned} &B\tilde{N}(\tilde{\mathbf{l}}\tilde{X},\tilde{\mathbf{l}}\tilde{Y}) = N(\mathbf{l}B\tilde{X},\mathbf{l}B\tilde{Y}) \\ &B\tilde{N}(\tilde{\mathbf{m}}\tilde{X},\tilde{\mathbf{m}}\tilde{Y}) = N(\mathbf{m}B\tilde{X},\mathbf{m}B\tilde{Y}) \\ &B\tilde{N}(\tilde{\mathbf{l}}\tilde{X},\tilde{\mathbf{l}}\tilde{Y}) = N(\mathbf{l}B\tilde{X},\mathbf{l}B\tilde{Y}) \\ &B\{\tilde{\mathbf{m}}\tilde{N}(\tilde{X},\tilde{Y}) = \mathbf{m}N(B\tilde{X},B\tilde{Y}). \end{aligned}$$

If $\tilde{L}$ and $\tilde{M}$ denote the complementary distributions corresponding to the projection operators $\tilde{\mathbf{l}}$ and $\tilde{\mathbf{m}}$ in $\mathcal{N}^p$, then, in view of the integrability conditions of the $\tilde{f}(3,\varepsilon)$-structure, we can state the following theorems.

Theorem 3.2. *If L is integrable in $\mathcal{M}^n$, then $\tilde{L}$ is also integrable in $\mathcal{N}^p$. If M is integrable in $\mathcal{M}^n$, then $\tilde{M}$ is also integrable in $\mathcal{N}^p$.*

Theorem 3.3. *If L and M both be integrable in $\mathcal{M}^n$ then $\tilde{L}$ and $\tilde{M}$ are integrable in $\mathcal{N}^p$.*

Theorem 3.4. *If the $f(3,\varepsilon)$-structure is integrable in $\mathcal{M}^n$ then the induced $\tilde{f}(3,\varepsilon)$-structure is also integrable.*

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