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SOME CLASSES OF COLOMBEAU'S GENERALIZED RANDOM PROCESSES

Zagorka Lozanov Crvenković, Stevan Pilipović
Institute of Mathematics, Faculty of Science, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Various types of random processes within Colombeau's generalized functions are introduced and analyzed. Stationary Colombeau's generalized random processes are defined, and for the processes determined by the Schwartz generalized random processes the explicit form of their correlation function is given. Further, a characterization of Colombeau's generalized processes with independent values is given if they are determined by the Schwartz generalized random processes.

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1. Introduction

Nonlinear stochastic differential equations in the frame of Colombeau's generalized functions were studied by F. Russo, [1], and M. Oberguggenberger, [6]. Their investigations, as well as the known monograph of Gel'fand and Vilenkin, have motivated us to consider Colombeau's generalized random processes.

We follow the systematic approach to the Schwartz generalized random processes given in [3], Chapters 3 and 4. We introduce and analyze various types of random processes within Colombeau's generalized functions. Our investigations are formulated in terms of nets of smooth stochastic processes $g(\varphi_\varepsilon, t)$, $\varepsilon > 0$, $t \in T \subset \mathbf{R}$, elements of spaces $\mathcal{E}_M(T,\Omega)$ or $\mathcal{E}_M(T, L^p(\Omega))$, $p \geq 1$, which, together with their derivatives have to satisfy power estimates uniformly on compact sets of T, and, which are identified whenever their difference $f_\varepsilon - g_\varepsilon$, $\varepsilon \in (0,1)$ belongs to an appropriate set, being an ideal of $\mathcal{E}_M(T,\Omega)$ or $\mathcal{E}_M(T, L^p(\Omega))$, $p \geq 1$. The classes of equivalence are called Colombeau's generalized random processes.

After giving the definitions of spaces $\mathcal{E}_M(T)$, $\mathcal{N}(T,\Omega)$ and $\mathcal{G}(T,\Omega)$, ([6]), we follow [1] and define the space $\mathcal{G}(T, L^p(\Omega))$, $p \geq 1$, and its subspace $\tilde{\mathcal{G}}(T, L^p(\Omega))$, $p \geq 1$. This enables us to define the expectation and the correlation function for elements in $\tilde{\mathcal{G}}(T, L^2(\Omega))$, study Gaussian Colombeau's generalized random processes, and give also a condition for the existence of such process. We define stationary Colombeau's generalized random processes, and for the processes determined by the Schwartz generalized random processes we find the explicit form of their correlation function. Further, we define Colombeau's generalized processes with independent values and give a characterization of such processes if they are determined by the Schwartz generalized random processes.

We should point out that this is probably one of the first papers in this domain, and that many problems remained unsolved. In fact, the space of Colombeau's generalized random processes is much larger than the space of the Schwartz generalized random processes, and the variety of new problems come into question. For example, a good version of the Bochner-Schwartz theorem for positive definite Colombeau's generalized functions is needed, as well as some theorems which would replace the representation theorems appearing in the Schwartz theory.

2. Basic notions

Let T be an open subset of $\mathbf{R}$ and $C_0^\infty(T)$ space of complex valued functions defined on $\mathbf{R}$ with compact supports contained in T. Denote $\mathcal{A}_0(\mathbf{R}) = \{\varphi \in C_0^\infty(\mathbf{R}) | \int \varphi(x)dx = 1\}$, and for $q \in \mathbf{N}_0$, $\mathcal{A}_q(\mathbf{R}) = \{\varphi \in \mathcal{A}_0 : \int x^j \varphi(x)dx = 0,\ 0 < j < q\}$; $A_q(\mathbf{R}^n) = \{\phi \in C_0^\infty(\mathbf{R}^n) : \phi(x_1, ...x_n) = \prod_{i=1}^n \varphi(x_i),\ \varphi \in \mathcal{A}_0(\mathbf{R})\}$. Put $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, $\check{\varphi}(x) = \varphi(-x)$, $x \in \mathbf{R}$. Here, we shall use the spaces $A_q(\mathbf{R}^2)$ and we shall denote $\phi(x,y) = \varphi(x)\varphi(y)$.

The basic space $\mathcal{E}(T)$ consists of all the functions $g : \mathcal{A}_0(\mathbf{R}) \to C^\infty(T)$. It is an algebra with the pointwise multiplication. More important is the subalgebra of the moderate elements $\mathcal{E}_M(T)$:

$$\begin{aligned}\mathcal{E}_M(T) = \; & \{g(\varphi_\varepsilon, x) \in \mathcal{E}(T) |\; (\forall K \subset\subset T) \\ & (\forall \alpha \in \mathbf{N}_0^n)\; (\exists N \in \mathbf{N}_0)\; (\forall \varphi \in \mathcal{A}_N(\mathbf{R})) \\ & (sup_{x\in K} |\partial^\alpha g(\varphi_\varepsilon, x)| = \mathcal{O}(\varepsilon^{-N}))\; \}\end{aligned}$$

Denote by Γ the set of sequences $\{a_q\}$ with positive elements which strictly increase to infinity. Then, the set of null elements $\mathcal{N}(T)$ in $\mathcal{E}(T)$ is defined as follows:

$$\begin{aligned}\mathcal{N}(T) = \; & \{g \in \mathcal{E}(T) |\; (\forall K \subset\subset T) \\ & (\forall \alpha \in \mathbf{N}_0^n)\; (\exists N \in \mathbf{N}_0)\; (\exists \{a_q\} \in \Gamma)\; (\forall q \geq N)\; (\forall \varphi \in \mathcal{A}_q(\mathbf{R})) \\ & (sup_{x\in K} |\partial^\alpha g(\varphi_\varepsilon, x)| \; = \mathcal{O}(\varepsilon^{a_q - N}))\; \}.\end{aligned}$$

Then,

$$\mathcal{G}(T) = \mathcal{E}_M(T)/\mathcal{N}(T).$$

The elements $G = [g] \in \mathcal{G}(T)$ are classes of equivalence represented by the elements $g \in \mathcal{E}_M(T)$. The pointwise product, the addition, and the derivation in $\mathcal{G}(T)$ are naturally defined by the corresponding representatives.

The space of generalized complex numbers is defined as $\bar{\mathbf{C}} = \mathcal{E}_c/\mathcal{F}_c$, where

$$\mathcal{E}_c = \{z : \mathcal{A}_0(\mathbf{R}) \to \mathbf{C} |\; (\exists N \in \mathbf{N})\; (\forall \varphi \in \mathcal{A}_N)(\mathbf{R})\;\; (|z(\varphi_\varepsilon)| \leq \mathcal{O}(\varepsilon^{-N}))\}.$$

$$\begin{aligned}\mathcal{F}_c = \; & \{z \in \mathcal{E}_c |\; (\exists N \in \mathbf{N})\; (\exists \{a_q\} \in \Gamma)\; (\forall \varphi \in \mathcal{A}_q(\mathbf{R}))\; (q \geq N) \\ & (|z(\varphi_\varepsilon)| \leq \mathcal{O}(\varepsilon^{a_q - N}))\; \}.\end{aligned}$$

The space of generalized real numbers $\bar{\mathbf{R}}$ is defined in an appropriate way by considering the above $z : \mathcal{A}_0(\mathbf{R}) \to \mathbf{R}$. It is the subspace of $\bar{\mathbf{C}}$.

The embedding of Schwartz's space $\mathcal{D}'$ into $\mathcal{G}(T)$, $Cd : \mathcal{D}'(T) \to \mathcal{G}(T)$ is defined in the following way. Denote by κ_ε a sequence of smooth functions on $C^\infty(T)$ such that $\kappa_\varepsilon \geq 0$, $\kappa_\varepsilon(x) = 0$ for $x \in \{x : \; d(x, CT)\} \leq \varepsilon$ $\kappa_\varepsilon(x) = 1$ for $x \in \{x : \; d(x, CT)\} \geq 2\varepsilon$.

Let $g \in \mathcal{D}'(T)$. Then $Cd(g) = [g(\varphi_\varepsilon, x)]$, where

$$g(\varphi_\varepsilon, x) = (g\kappa_\varepsilon * \varphi_\varepsilon)(x) = < g(t), \varphi_\varepsilon(x - t) >, \; x \in \mathbf{R}, \;\; \varphi_\varepsilon \in \mathcal{A}_0(\mathbf{R}).$$

This is an injective mapping.

Let K be a compact set in $\mathbf{R}$ and $G = [g(\varphi_\varepsilon, x)] \in \mathcal{G}(T)$. The integral $\int_K G(x)dx$ is defined by its representative $\int_K g(\varphi_\varepsilon, x)dx$.

In [5] we gave the definitions of positive and positive definite Colombeau's generalized function in the following way.

Definition 1. *Let $G \in \mathcal{G}(\mathbf{R})$ and I be an interval of $\mathbf{R}$. Then G is positive, $G \geq 0$, (resp. negative $G \leq 0$,) on I if it has a representative $g(\varphi_\varepsilon, x) \in \mathcal{E}_M(\mathbf{R})$ such that*

$$(1) \quad \begin{array}{l}(\forall\varphi \in \mathcal{A}_0(\mathbf{R}))\ (\exists\eta > 0)\\ ((x \in I)\ (\varepsilon < \eta)) \Rightarrow (g(\varphi_\varepsilon, x) \geq 0) \quad (\textit{resp. } g(\varphi_\varepsilon, x) \leq 0)\end{array}$$

Definition 2. *Let $G \in \mathcal{G}(\mathbf{R})$. Then G is positive definite if it has a representative $g(\varphi_\varepsilon, x) \in \mathcal{E}_M$ such that*

$$(2) \quad \begin{array}{l}(\forall\varphi \in \mathcal{A}_0(\mathbf{R}))\ (\exists\eta > 0)\ (\forall x_1, x_2, ..., x_n \in \mathbf{R})\ (\forall z_1, z_2, ..., z_n \in C)\\ (\varepsilon < \eta) \Rightarrow (\sum_{i=1}^n g(\varphi_\varepsilon, x_i - x_j)z_i\bar{z}_j \geq 0).\end{array}$$

We can add here that in [5] we gave more general definitions of these notions, in order to have the coherence with the notions of positive distributions and positive definite distributions. Definitions 1 and 2 are converient for the purposes of this paper.

3. Colombeau's generalized random processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be an open interval of real line.

Definition 3. *A map $Y : \Omega \to \mathcal{G}(T)$ is a random Colombeau's generalized function if there is $g_Y : \Omega \times \mathcal{A}_0(\mathbf{R}) \times T \to \mathbf{C}$ such that*

(i) $g_Y(\omega, \cdot)$ represents $Y(\omega)$ for almost every $\omega \in \Omega$,

(ii) For every $\varphi \in \mathcal{A}_0(\mathbf{R})$, $\Omega \times T \ni (\omega, t) \mapsto g_Y(\omega, \varphi, t) \in \mathbf{C}$ is a measurable process.

The above definition (given in [1] and [6]) can be adapted to the vector valued case $L^p(\Omega)$, (see [1], [6]) as follows.

By $\mathcal{E}(T, L^p(\Omega))$, $p \geq 1$ is denoted a space of all functions $g : \mathcal{A}_0(\mathbf{R}) \times T \to L^p(\Omega)$ such that $t \mapsto g(\varphi_\varepsilon, t)$ belong to $C^\infty(T)$ for each fixed $\varphi \in \mathcal{A}_0(\mathbf{R})$. Then, spaces of moderate functions, null functions, and generalized Colombeau's random processes (C grp) are defined respectively by:

$$\begin{aligned}\mathcal{E}_M(T, L^p(\Omega)) = \quad & \{g(\omega, \varphi_\varepsilon, t) \in \mathcal{E}(T, L^p(\Omega)) | (\forall K \subset\subset T)\ (\forall \alpha \in \mathbf{N}_0^n) \\ & (\exists N \in \mathbf{N}_0)\ (\forall \varphi \in \mathcal{A}_N(\mathbf{R})) \\ & (\sup_{t \in K} ||\partial^\alpha g(\varphi_\varepsilon, t)||_p = \mathcal{O}(\varepsilon^{-N}))\}.\end{aligned}$$

The set of null elements is:

$$\begin{aligned}\mathcal{N}(T, L^p(\Omega)) = \quad & \{g(\omega, \varphi_\varepsilon, t) \in \mathcal{E}(T, L^p(\Omega)) | (\forall K \subset\subset T)\ (\forall \alpha \in \mathbf{N}_0^n) \\ & (\exists N \in \mathbf{N}_0)\ (\exists \{a_q\} \in \Gamma)\ (\forall q \geq N)\ (\forall \varphi \in \mathcal{A}_q(\mathbf{R})) \\ & (\sup_{t \in K} ||\partial^\alpha g(\varphi_\varepsilon, t)||_p = \mathcal{O}(\varepsilon^{a_q - N}))\}.\end{aligned}$$

$$\mathcal{G}(T, L^p(\Omega)) = \mathcal{E}_M(T, L^p(\Omega)) / \mathcal{N}(T, L^p(\Omega)).$$

Since $\mathcal{E}(T, L^p(\Omega))$ is not an algebra under the multiplication, we shall consider a subspace of $\mathcal{G}(T, L^p(\Omega))$, (see [1]).

Definition 4. *It is said that $Y \in \mathcal{G}(T, L^p(\Omega))$ has a samplewise version [1] if there is a representative of Y, g_Y, called the modification of Y, which is a Colombeau's generalized function. A subspace of $\mathcal{G}(T, L^p(\Omega))$ with elements having samplewise version will be denoted by $\tilde{\mathcal{G}}(T, L^p(\Omega))$.*

Clearly, $\tilde{\mathcal{G}}(T, L^p(\Omega))$ is an algebra. If $Y \in \tilde{\mathcal{G}}(T, L^p(\Omega))$, then $\partial^\alpha Y \in \tilde{\mathcal{G}}(T, L^p(\Omega))$, $\alpha \in \mathbf{N}_0^n$, see [4].

The spaces of Colombeau's generalized random variables $\mathcal{G}(L^p(\Omega))$ and $\tilde{\mathcal{G}}(L^p(\Omega))$ can be defined in an appropriate way, [6]. If Y is in $\tilde{\mathcal{G}}(T, L^p(\Omega))$ with the representative $g_Y(\omega, \varphi_\varepsilon, t)$, then $Y(t_0)$, with the representative $g_Y(\omega, \varphi_\varepsilon, t_0)$, is a generalized random variable.

We shall use the definitions of mathematical expectation, correlation function, and covariance function of elements in $\mathcal{G}(L^p(T, \Omega))$ given in [4].

Definition 5. *Let $Y \in \tilde{\mathcal{G}}(T, L^1(\Omega))$ with the modification g_Y. The mathematical expectation of Y, $E(Y)$, is an element of $\mathcal{G}(T)$ represented by*

$$m_Y(\varphi_\varepsilon, t) = E(g_Y(\varphi_\varepsilon, t)).$$

Definition 6. *Let $Y \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with modification g_Y.*

1. *Correlation function of Y, $B(Y)$, is an element of $\mathcal{G}(T^2)$ represented by*

$$B_Y(\phi_\varepsilon, t, s) = E(g_Y(\varphi_\varepsilon, t) g_Y(\varphi_\varepsilon, s)), \quad \phi_\varepsilon(t,s) = \varphi_\varepsilon(t)\varphi_\varepsilon(s).$$

2. *Covariance function of Y, $C(Y)$, is an element of $\mathcal{G}(T^2)$ represented by*

$$C_Y(\phi_\varepsilon, t, s) = B_Y(\phi_\varepsilon, t, s) - m_Y(\varphi_\varepsilon, t) m_Y(\varphi_\varepsilon, s)$$

The following assertion was given in [4]

Theorem 1. *Let $Y \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with a modification g_Y such that $\forall K \subset\subset T$, $\exists C_K \in L^2(\Omega)$, $C_K \geq 0$, such that $sup_{t \in K} |g'_Y(\omega, \varphi_\varepsilon, t)| \leq C_K(\omega)$. Then*

$$\frac{\partial}{\partial t} m_Y(\varphi_\varepsilon, t) = E(\frac{\partial}{\partial t} g_Y(\varphi_\varepsilon, t)), \tag{3}$$

$$\frac{\partial^2}{\partial t \partial s} B_Y(\phi_\varepsilon, t, s) = E(\frac{\partial}{\partial t}(g_Y(\varphi_\varepsilon, t) \frac{\partial}{\partial s} g_Y(\varphi_\varepsilon, s))). \tag{4}$$

Let $Y(t)$ be an ordinary $L^2(\Omega)$ process with continuous sample functions. The corresponding element in $\tilde{\mathcal{G}}(T, L^2(\Omega))$, $Cd(Y)$ is represented by

$$(5)\ g_Y(\omega, \varphi_\varepsilon, t) = (Y(\omega, \cdot) * \varphi_\varepsilon(\cdot))(t) = \int_T Y(\omega, u)\varphi_\varepsilon(t-u)du, \quad \varphi_\varepsilon \in \mathcal{A}_0.$$

Proposition 1. *Let $Y(t)$ be a $L^2(\Omega)$ process with continuous sample functions, with mathematical expectation $m(t)$ and correlation function $B(t,s)$. Let $g_Y(\omega, \varphi_\varepsilon, t)$ be a representative for $Cd(Y)$ given by (5). Then*

(i) $m_{Cd(Y)}(\varphi_\varepsilon, t) = (m * \varphi_\varepsilon)(t)$,

(ii) $B_{Cd(Y)}(\phi_\varepsilon, t, s) = (B * \phi_\varepsilon)(t, s)$.

Proof. The proof follows from the known properties of the integral of a process with continuous sample functions:

$$m_{Cd(Y)}(\varphi_\varepsilon, t) = E(\int_T Y(\omega, u)\varphi_\varepsilon(t-u)du) =$$

$$\int_T E(Y(\omega,u))\varphi_\varepsilon(t-u)du = (m * \varphi_\varepsilon)(t).$$

The same is valid if Y is a generalized random process in the sense of Gel'fand, [3], i.e. if the sample functions is in $\mathcal{D}'$.

3.1 Gaussian Colombeau's generalized random process

Definition 7. *Let $Y \in \tilde{\mathcal{G}}(T, L^2(\Omega))$. It is said that Y is Gaussian Colombeau's grp (GC grp) if for any $t_1, ..., t_n \in \mathbf{R}$ and any $\varepsilon > 0$, $\phi_\varepsilon(x_1, ..., x_n) = \varphi_\varepsilon(x_1)\cdots\varphi_\varepsilon(x_n) \in \mathcal{A}_0(\mathbf{R}^n)$, $\varphi_\varepsilon \in \mathcal{A}_0(\mathbf{R})$, the probability that the random variable*
$\mathbf{Y} = (g_Y(\varphi, t_1), ..., g_Y(\varphi, t_n))$ belongs to a Borel set S, is

$$P_{\phi_\varepsilon, t_1, ...t_n}\{\mathbf{Y} \in S\} = \int_S \frac{\sqrt{det\Lambda}}{(2\pi)^{n/2}} exp\{-\frac{1}{2}(\Lambda u, u)\}du, \tag{6}$$

where Λ is a non-degenerate positive definite matrix, and

$$(\Lambda u, u) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij}u_i u_j.$$

In [4] we proved two following theorems.

Theorem 2. *The derivatives of a Gaussian Colombeau's generalized process Y is again a Gaussian Colombeau's generalized process.*

Theorem 3. *Let Y be a GC grp. Then for any $\phi \in \mathcal{A}_0(\mathbf{R}^n)$ and $t_1, ..., t_n \in \mathbf{R}$,*

$$\Lambda = ||B_Y(\phi_\varepsilon, t_i, t_j)||^{-1}. \tag{7}$$

The following theorem completes the characterization of GC grp.

Theorem 4. *Let $[m(\varphi_\varepsilon, t)] \in \mathcal{G}(T)$ and $[B(\phi_\varepsilon, t, s)] \in \mathcal{G}(T^2)$, $\phi_\varepsilon(t,s) = \varphi_\varepsilon(t)\varphi_\varepsilon(s)$, be such that the covariance function $C(Y) \in \mathcal{G}(T^2)$, represented by*

$$C(\phi_\varepsilon, t, s) = B(\phi_\varepsilon, t, s) - m(\varphi_\varepsilon, t)m(\varphi_\varepsilon, s)$$

is positive definite. Then, there is a Gaussian Colombeau's grp Y with the representative g_Y for which

$$m_Y(\varphi_\varepsilon, t) = m(\varphi_\varepsilon, t),$$

$$C_Y(\phi_\varepsilon, t, s) = C(\phi_\varepsilon, t, s).$$

Proof. Since

$$C(\phi_\varepsilon, t, s) = B(\phi_\varepsilon, t, s) - m(\varphi_\varepsilon, t)m(\varphi_\varepsilon, s)$$

is positive definite, for any finite set $t_1, ..., t_n$ in T and $\varphi \in \mathcal{A}_0$ an $n-$dimensional Gaussian distribution with the mean values $m(\varphi_\varepsilon, t_1)$, ..., $m(\varphi_\varepsilon, t_n)$ and covariance matrix $||C(\phi_\varepsilon, t_i, t_j)||$ is defined. This is the distribution function with the characteristic function

$$exp\{-\frac{1}{2}\sum_{i,j=1}^{n} C(\phi_\varepsilon, t_i, t_j)\lambda_i\lambda_j + i\sum_{i=1}^{n} \lambda_i m(\varphi_\varepsilon, t_i)\} \tag{8}$$

Moreover, if $k \leq n$, the distribution defined by the characteristic function (8) assigns to $g_Y(\varphi_\varepsilon, t_1), ... \, g_Y(\varphi_\varepsilon, t_k)$ a Gaussian distribution by means $m(\varphi_\varepsilon, t_1)$, ..., $m(\varphi_\varepsilon, t_k)$ and covariance matrix $||C(\phi_\varepsilon, t_i, t_j)|| \; i, j \leq k$. Thus, if $k < n$ the marginal distribution of $(g_Y(\varphi_\varepsilon, t_1), \, ..., g_Y(\varphi_\varepsilon, t_k))$ is the same as assigned to $g_Y(\varphi_\varepsilon t_1), \, ..., g_Y(\varphi_\varepsilon, t_k)$. Consequently, the consistency conditions of Kolmogorov are satisfied, where for the space of sample functions we take the space of Colombeau's functions.

Corollary 1. *Let $Y \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with the mean $m_Y(\varphi_\varepsilon, t) \in \mathcal{G}(T)$ and correlation function $B_Y(\phi_\varepsilon, t, s) \in \mathcal{G}(T^2)$. Then, there is a GC grp with the same mean and correlation function.*

4. Stationary Colombeau's grp.

Definition 8. *$Y \in \tilde{\mathcal{G}}(T, L^p(\Omega))$, $p > 0$, is a stationary C grp if it has a modification $g_Y(\varphi_\varepsilon, \omega, t)$ with the following property. For every $\varphi \in \mathcal{A}_0(\mathbf{R})$, $t_1, ..., t_n \in T$, $h \in \mathbf{R}$ and every $\varepsilon > 0$, random variables $(g_Y(\varphi_\varepsilon, t_1 + h)$, ..., $g_Y(\varphi_\varepsilon, t_n + h))$ and $(g_Y(\varphi_\varepsilon, t_1)$, ..., $g_Y(\varphi_\varepsilon, t_n))$ are identically distributed.*

Proposition 2. *Let $Y = [g_Y] \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ be a stationary C grp. Then* *(i) The mathematical expectation of* Y *is a generalized number.*

(ii) If $B_Y(\phi_\varepsilon, t, s) \in \mathcal{G}(T^2)$ *is a representative of the correlation function of* Y*, then there exists a positive definite generalized function* $B(\varphi_\varepsilon, t) \in \mathcal{G}(T)$ *such that for every* $t, s \in T$

$$B_Y(\phi_\varepsilon t, s) = B(\varphi_\varepsilon, t - s).$$

Proof. *(i)* We know that $E(g_Y(\varphi_\varepsilon, t))$ is in $\mathcal{E}_M(T)$. Since for every fixed $\varepsilon > 0$ $g_Y(\varphi_\varepsilon, t)$ is a stationary process with smooth sample functions, we have

$$m(\varphi_\varepsilon, t + h) = E(g_Y(\varphi_\varepsilon, t + h)) = E(g_Y(\varphi_\varepsilon, t)) = m(\varphi_\varepsilon, t),$$

it follows that $m_Y(\varphi_\varepsilon, t) = m_Y(\varphi_\varepsilon)$, $t \in T$. Thus, $[m_Y(\varphi_\varepsilon, t)] \in \bar{\mathbf{R}}$.
(ii) Since Y is stationary, for every $h \in \mathbf{R}$ $t, s \in T$, and $\phi_\varepsilon(x, y) = \varphi_\varepsilon(x)\varphi_\varepsilon(y)$, we have

$$B_Y(\varphi_\varepsilon, t, s) = B_Y(\varphi_\varepsilon, t + h, s + h).$$

Thus, for every fixed ϕ and ε we have

$$B_Y(\phi_\varepsilon, t, s) = B(\varphi_\varepsilon, t - s).$$

Since for every $\varepsilon > 0$ B_Y is positive definite smooth function, it follows that B represents a positive definite Colombeau's generalized function.

Proposition 3. *Let* $Y(t)$ $t \in T$*, be an* $L^2(\Omega)$ *stationary process with continuous sample functions, with the mathematical expectation* $m(t) = a$*, and the correlation function* $B(\tau)$*. Then, for every* $\varphi \in \mathcal{A}_0$*,*

(i) $m_{Cd(Y)}(\varphi_\varepsilon, t) = a \int_T \varphi_\varepsilon(x)dx.$

(ii)

$$B_{Cd(Y)}(\phi_\varepsilon, t, s) = B_{Cd(Y)}(\phi_\varepsilon, \tau) = (B * (\varphi_\varepsilon * \varphi_\varepsilon))(\tau),$$

$\tau = t - s$, $\phi_\varepsilon(x, y) = \varphi_\varepsilon(x)\partial_\varepsilon(y)$.

Proof. *(i)* Proposition 1 implies

$$m_{Cd(Y)}(\varphi_\varepsilon, t) = (m * \varphi_\varepsilon)(t) = a \int_T \varphi_\varepsilon(u)du.$$

(ii)

$$B_{Cd(Y)}(\phi_\varepsilon t, s) = B * \phi_\varepsilon(t,s) = \int_T \int_T B(u,v)\varphi_\varepsilon(u-s)\varphi_\varepsilon(v-t)dudv =$$

$$\int_T \int_T B(s+u, t+v)\varphi_\varepsilon(u)\varphi_\varepsilon(vt)dudv =$$

$$\int_T \int_T B(s+u-(t+v))\varphi_\varepsilon(u)\varphi_\varepsilon(v)dudv =$$

$$\int_T \int_T B(s-t+u-v))\varphi_\varepsilon(u)\varphi_\varepsilon(v)dudv =$$

$$\int_T \int_T B(\tau+h))\varphi_\varepsilon(v+hs)\varphi_\varepsilon(v)dudh =$$

$$(B*(\varphi_\varepsilon * \varphi_\varepsilon))(\tau),$$

where $s-t=\tau$, and $u-v=h$.

5. C grp with independent values

In [3], a generalized random process $\Phi(\varphi)$, $\varphi \in \mathcal{D}(T)$ is defined to have independent values at every point if the random variables $\Phi(\varphi_1)$ and $\Phi(\varphi_2)$ are independent, whenever $\varphi_1(x)\varphi_2(x) = 0$.

Definition 9. *Let $Y \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with modification g_Y. We say that it has independent values at every point of T if for every $t_1 \neq t_2$ and every $\varphi \in \mathcal{A}_0$ exists $\varepsilon_0 > 0$ such that $g_Y(\varphi_\varepsilon, t_1)$ and $g_Y(\varphi_\varepsilon, t_2)$ are independent random variables for $\varepsilon < \varepsilon_0$.*

Theorem 5. *Let $f \in \mathcal{L}(\mathcal{D}(T), L^2(\Omega))$. Then f has independent values at every point in the sense of Gel'fand and Vilenkin if and only if $(f * \varphi_\varepsilon)$ has independent values at every point of T.*

Proof. Let f has independent values at every point of T. Let $t_1 \neq t_2$, $t_1, t_2 \in T$. Since

$$(f * \varphi_\varepsilon)(t_1) = \int_T f(u)\varphi_\varepsilon(t_j - u)du, \ \ j = 1, 2,$$

we have that for some $\varepsilon_0 > 0$ and $\varepsilon < \varepsilon_0$ $\sup \varphi_\varepsilon(t_1 - u) \cap \sup \varphi_\varepsilon(t_2 - u) = \emptyset$, and thus $(f * \varphi_\varepsilon)(t_1)$ and $(f * \varphi_\varepsilon)(t_2)$ have independent values.

Conversely, if $(f * \varphi_\varepsilon)(t_1)$ and $(f * \varphi_\varepsilon)(t_2)$ have independent values, and if $\varphi_1, \varphi_2 \in \mathcal{D}(T)$ are such that $\sup \varphi_1 \cap \sup \varphi_2 = \emptyset$, we have

$$< f, \varphi_j > = \lim_{\varepsilon \to 0} \int_T (f * \varphi_\varepsilon) \varphi_j(u) du, \quad j = 1, 2 \tag{9}$$

There exists $\varepsilon_0 > 0$ such that for every fixed $\varepsilon < \varepsilon_0$, $(f * \varphi_\varepsilon)(t)\varphi_1(t)$ and $(f * \varphi_\varepsilon)(t)\varphi_2(t)$ are independent for every $t \in T$ and for every fixed $\varepsilon < \varepsilon_0$, the same holds for the integrals in (9). Now, letting $\varepsilon \to 0$, we obtain that $< f, \varphi_1 >$ and $< f, \varphi_2 >$ are independent.

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