NOVI SAD J. MATH.
VOL. 27, NO. 2, 1997, 1-12

GENERALIZATIONS OF COMMUTATIVE NEUTRIX CONVOLUTION PRODUCTS OF FUNCTIONS

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Abstract

The commutative neutrix convolution product of the functions $x^r e^{\lambda_1} \cos_-(\lambda_2 x)$ and $x^s e^{\mu_1} \cos_+(\mu_2 x)$ is evaluated. Further, similar commutative neutrix convolution products are then evaluated.

AMS Mathematics Subject Classification (1991): Primary 46F10, Secondary 44A35
Key words and phrases: distribution, neutrix, neutrix limit, neutrix convolution product

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side (see Gel'fand and Shilov [8]).

Note that if f and g are locally summable functions satisfying either of the above conditions, then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

This definition of the convolution product is rather restrictive, and so a neutrix convolution product was introduced in [4]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \le \tau(x) \le 1$,

(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,

(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu, \end{cases}$$

for $\nu > 0$.

We now give a new neutrix convolution product which generalizes the one given in [4].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ and $g_\nu = g\tau_\nu$ for $\nu > 0$. Then the neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_\nu * g_\nu\}$, provided that the limit h exists in the sense that*

$$\mathop{\mathrm{N-lim}}_{\nu\to\infty} \langle f_\nu * g_\nu, \varphi\rangle = \langle h, \varphi\rangle,$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput [1]), having domain N' the positive reals and range N'' the complex numbers, with negligible functions finite linear sums of the functions

$$\nu^\lambda \ln^{r-1}\nu,\ \ln^r \nu,\ \nu^{r-1}e^{\mu\nu}, \quad (\text{real } \lambda > 0,\ \text{complex } \mu \ne 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g_\nu$ is defined in Gel'fand and Shilov's sense, the distributions f_ν and g_ν both having bounded support.

In the original definition of the neutrix convolution product, the domain of the neutrix N was the set of positive integers $N' = \{1, 2, \ldots, n, \ldots\}$, the range was the set of real numbers and the negligible functions were finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all funtions which converge to zero in the usual sense as n tends to infinity. In [2], the set of negligible functions was extended to include finite linear sums of the functions

$$n^\lambda e^{\mu n} \quad (\mu > 0).$$

In [3], the domain of the neutrix N was replaced by the set of real numbers and the set of negligible functions was extended to include finite linear sums of the functions

$$\nu^\mu \cos \lambda\nu,\ \nu^\mu \sin \lambda\nu \quad (\lambda \neq 0).$$

However, using complex λ, these negligible functions are not included in the above definition.

It is easily seen that any results proved with the earlier definitions hold with this latest definition. The following theorem, proved in [4], therefore holds, showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \boxast g$ exists and*

$$f \boxast g = f * g.$$

We now define the locally summable functions $e_+^{\lambda x}$, $e_-^{\lambda x}$, $\cosh_+(\lambda x)$, $\cosh_-(\lambda x)$, $\sinh_+(\lambda x)$ and $\sinh_-(\lambda x)$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0, \end{cases}$$

$$\cosh_+(\lambda x) = \tfrac{1}{2}[e_+^{\lambda x} + e_+^{-\lambda x}], \quad \cosh_-(\lambda x) = \tfrac{1}{2}[e_-^{\lambda x} + e_-^{-\lambda x}],$$
$$\sinh_+(\lambda x) = \tfrac{1}{2}[e_+^{\lambda x} - e_+^{-\lambda x}], \quad \sinh_-(\lambda x) = \tfrac{1}{2}[e_-^{\lambda x} - e_-^{-\lambda x}].$$

It follows that

$$\cosh_-(\lambda x) + \cosh_+(\lambda x) = \cosh(\lambda x), \quad \sinh_-(\lambda x) + \sinh_+(\lambda x) = \sinh(\lambda x).$$

The following theorem was proved in [2].

Theorem 2. *The neutrix convolution product* $(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x})$ *exists and*

$$e_-^{\lambda x} \boxast e_+^{\mu x} = \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}, \tag{1}$$

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} \tag{2}$$

where $D_\lambda = \partial/\partial\lambda$ *and* $D_\mu = \partial/\partial\mu$, *for* $\lambda \neq \mu$ *and* $r, s = 0, 1, 2, \dots$, *these neutrix convolution products existing as convolution products if* $\lambda > \mu$ *(or* $\Re\lambda > \Re\mu$ *for complex* λ, μ*) and*

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\lambda x}) = -B(r+1, s+1)\,\mathrm{sgn}\, x . x^{r+s+1} e^{\lambda x}, \tag{3}$$

where B *denotes the Beta function, for all* λ *and* $r, s = 0, 1, 2, \dots$.

We note for future reference that if $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$, then the right-hand side of equation (2) can be replaced by

$$D_{\lambda_1}^r D_{\mu_1}^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}. \tag{4}$$

The next theorem is an immediate consequence of Theorem 2, on noting that the neutrix convolution product is distributive with respect to addition.

Theorem 3. *The neutrix convolution products*

$$[x^r e^{\lambda_1 x} \cosh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x} \cosh_+(\mu_2 x)],$$
$$[x^r e^{\lambda_1 x} \sinh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x} \sinh_+(\mu_2 x)],$$
$$[x^r e^{\lambda_1 x} \cosh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x} \sinh_+(\mu_2 x)],$$
$$[x^r e^{\lambda_1 x} \sinh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x} \cosh_+(\mu_2 x)]$$

exist for all $\lambda_1, \lambda_2, \mu_1, \mu_2$ *and* $r, s = 0, 1, 2, \dots$.

It is of interest to obtain explicit formulae for these neutrix convolution products. For example, $[x^r e^{\lambda_1 x}\cosh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x}\cosh_+(\mu_2 x)]$ may be evaluated as follows. First of all, using equation (1) with $(\lambda_1-\mu_1)^2 \neq (\lambda_2 \pm \mu_2)^2$, we have

$$\begin{aligned}
4[e^{\lambda_1 x}&\cosh_-(\lambda_2 x)] \boxast [e^{\mu_1 x}\cosh_+(\mu_2 x)] = \\
&= [e^{\lambda_1 x}(e_-^{\lambda_2 x}+e_-^{-\lambda_2 x})] \boxast [e^{\mu_1 x}(e_+^{\mu_2 x}+e_+^{-\mu_2 x})] \\
&= e_-^{(\lambda_1+\lambda_2)x} \boxast e_+^{(\mu_1+\mu_2)x} + e_-^{(\lambda_1-\lambda_2)x} \boxast e_+^{(\mu_1+\mu_2)x} + e_-^{(\lambda_1+\lambda_2)x} \boxast e_+^{(\mu_1-\mu_2)x} + \\
&\qquad + e_-^{(\lambda_1-\lambda_2)x} \boxast e_+^{(\mu_1-\mu_2)x} \\
&= \frac{e_+^{(\mu_1+\mu_2)x}+e_-^{(\lambda_1+\lambda_2)x}}{\lambda_1+\lambda_2-\mu_1-\mu_2} + \frac{e_+^{(\mu_1+\mu_2)x}+e_-^{(\lambda_1-\lambda_2)x}}{\lambda_1-\lambda_2-\mu_1-\mu_2} + \\
&\qquad + \frac{e_+^{(\mu_1-\mu_2)x}+e_-^{(\lambda_1+\lambda_2)x}}{\lambda_1+\lambda_2-\mu_1+\mu_2} + \frac{e_+^{(\mu_1-\mu_2)x}+e_-^{(\lambda_1-\lambda_2)x}}{\lambda_1-\lambda_2-\mu_1+\mu_2} \\
&= \frac{(\lambda_1-\mu_1)(e_+^{(\mu_1+\mu_2)x}+e_+^{(\mu_1-\mu_2)x}+e_-^{(\lambda_1+\lambda_2)x}+e_-^{(\lambda_1-\lambda_2)x})}{(\lambda_1+\lambda_2-\mu_1-\mu_2)(\lambda_1-\lambda_2-\mu_1+\mu_2)} + \\
&\quad + \frac{(\lambda_2-\mu_2)(e_+^{(\mu_1-\mu_2)x}-e_+^{(\mu_1+\mu_2)x}+e_-^{(\lambda_1-\lambda_2)x}-e_-^{(\lambda_1+\lambda_2)x})}{(\lambda_1+\lambda_2-\mu_1-\mu_2)(\lambda_1-\lambda_2-\mu_1+\mu_2)} + \\
&\quad + \frac{(\lambda_1-\mu_1)(e_+^{(\mu_1+\mu_2)x}+e_+^{(\mu_1-\mu_2)x}+e_-^{(\lambda_1+\lambda_2)x}+e_-^{(\lambda_1-\lambda_2)x})}{(\lambda_1-\lambda_2-\mu_1-\mu_2)(\lambda_1+\lambda_2-\mu_1+\mu_2)} + \\
&\quad + \frac{(\lambda_2+\mu_2)(e_+^{(\mu_1+\mu_2)x}-e_+^{(\mu_1-\mu_2)x}+e_-^{(\lambda_1-\lambda_2)x}-e_-^{(\lambda_1+\lambda_2)x})}{(\lambda_1-\lambda_2-\mu_1-\mu_2)(\lambda_1+\lambda_2-\mu_1+\mu_2)} \\
&= \frac{2(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2} + \\
&\qquad - \frac{2(\lambda_2-\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)+e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2} + \\
&\qquad + \frac{2(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2} + \\
&\qquad + \frac{2(\lambda_2+\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)-e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2},
\end{aligned}$$

giving $[e^{\lambda_1 x}\cosh_-(\lambda_2 x)] \boxast [e^{\mu_1 x}\cosh_+(\mu_2 x)]$. In particular, if $\lambda_1=\mu_1=\lambda$ and $\lambda_2 \neq \pm\mu_2$, we have

$$[e^{\lambda x}\cosh_-(\lambda_2 x)] \boxast [e^{\lambda x}\cosh_+(\mu_2 x)] \;=\; \frac{e^{\lambda x}[\sinh_+(\mu_2 x)+\sinh_-(\lambda_2 x)]}{\lambda_2-\mu_2} +$$

$$-\frac{e^{\lambda x}[\sinh_+(\mu_2 x)-\sinh_-(\lambda_2 x)]}{\lambda_2+\mu_2}$$
$$=\frac{2e^{\lambda x}[\lambda_2\sinh_-(\lambda_2 x)+\mu_2\sinh_+(\mu_2 x)]}{\lambda_2^2-\mu_2^2}.$$

More generally, it follows from equation (2), with the modified form (4) that

$$\begin{aligned}
&[x^r e^{\lambda_1 x}\cosh_-(\lambda_2 x)] \boxed{*} [x^s e^{\mu_1 x}\cosh_+(\mu_2 x)] = \\
&= D^r_{\lambda_1} D^s_{\mu_1}[e^{\lambda_1 x}\cosh_-(\lambda_2 x)] \boxed{*} [e^{\mu_1 x}\cosh_+(\mu_2 x)] \\
&= D^r_{\lambda_1} D^s_{\mu_1}\Big\{\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+ \\
&\quad -\frac{(\lambda_2-\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)+e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+ \\
&\quad +\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}+ \\
&\quad +\frac{(\lambda_2+\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)-e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}\Big\}.
\end{aligned} \tag{5}$$

Now suppose that $\lambda_1=\mu_1=\lambda$ and $\lambda_2=\mu_2=\mu\neq 0$. Then using equation (3) we have

$$\begin{aligned}
&4[x^r e^{\lambda x}\cosh_-(\mu x)] \boxed{*} [x^s e^{\lambda x}\cosh_+(\mu x)] = \\
&= [x^r e^{\lambda x}(e_-^{\mu x}+e_-^{-\mu x})] \boxed{*} [x^s e^{\lambda x}(e_+^{\mu x}+e_+^{-\mu x})] \\
&= (x^r e_-^{(\lambda+\mu)x}) \boxed{*} (x^s e_+^{(\lambda+\mu)x}) + (x^r e_-^{(\lambda+\mu)x}) \boxed{*} (x^s e_+^{(\lambda-\mu)x}) + \\
&\qquad +(x^r e_-^{(\lambda-\mu)x}) \boxed{*} (x^s e_+^{(\lambda+\mu)x}) + (x^r e_-^{(\lambda-\mu)x}) \boxed{*} (x^s e_+^{(\lambda-\mu)x}) \\
&= B(r+1,s+1)\operatorname{sgn} x.x^{r+s+1}(e^{(\lambda+\mu)x}+e^{(\lambda-\mu)x}) + \\
&\qquad +(x^r e_-^{(\lambda+\mu)x}) \boxed{*} (x^s e_+^{(\lambda-\mu)x}) + (x^r e_-^{(\lambda-\mu)x}) \boxed{*} (x^s e_+^{(\lambda+\mu)x}) \\
&= 2B(r+1,s+1)\operatorname{sgn} x.x^{r+s+1}e^{\lambda x}\cosh(\mu x) + \\
&\qquad +(x^r e_-^{(\lambda+\mu)x}) \boxed{*} (x^s e_+^{(\lambda-\mu)x}) + (x^r e_-^{(\lambda-\mu)x}) \boxed{*} (x^s e_+^{(\lambda+\mu)x}).
\end{aligned}$$

In order to evaluate

$$(x^r e_-^{(\lambda+\mu)x}) \boxed{*} (x^s e_+^{(\lambda-\mu)x}) + (x^r e_-^{(\lambda-\mu)x}) \boxed{*} (x^s e_+^{(\lambda+\mu)x}),$$

we consider

$$e_-^{(\lambda+\mu)x} \boxast e_+^{(\nu-\mu)x} + e_-^{(\lambda-\mu)x} \boxast e_+^{(\nu+\mu)x} = \frac{e_+^{(\nu-\mu)x} + e_-^{(\lambda+\mu)x}}{\lambda-\nu+2\mu} + \frac{e_+^{(\nu+\mu)x} + e_-^{(\lambda-\mu)x}}{\lambda-\nu-2\mu}$$

$$= \frac{(\lambda-\nu)(e_+^{(\nu-\mu)x} + e_+^{(\nu+\mu)x} + e_-^{(\lambda-\mu)x} + e_-^{(\lambda+\mu)x})}{(\lambda-\nu)^2-4\mu^2} + - \frac{2\mu(e_+^{(\nu-\mu)x} - e_+^{(\nu+\mu)x} - e_-^{(\lambda-\mu)x} + e_-^{(\lambda+\mu)x})}{(\lambda-\nu)^2-4\mu^2}$$

$$= \frac{2(\lambda-\nu)[e^{\nu x}\cosh_+(\mu x) + e^{\lambda x}\cosh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2} + + \frac{4\mu[e^{\nu x}\sinh_+(\mu x) - e^{\lambda x}\sinh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2},$$

on using equation (1). Then

$$(x^r e_-^{(\lambda+\mu)x}) \boxast (x^s e_+^{(\lambda-\mu)x}) + (x^r e_-^{(\lambda-\mu)x}) \boxast (x^s e_+^{(\lambda+\mu)x}) = = D_\lambda^r D_\nu^s [e_-^{(\lambda+\mu)x} \boxast e_+^{(\nu-\mu)x} + e_-^{(\lambda-\mu)x} \boxast e_+^{(\nu+\mu)x}]_{\nu=\lambda}$$

and so

$$[x^r e^{\lambda x}\cosh_-(\mu x)] \boxast [x^s e^{\lambda x}\cosh_+(\mu x)] = = \tfrac{1}{2} B(r+1,s+1) x^{r+s+1} e^{\lambda x}\cosh(\mu x) + +D_\lambda^r D_\nu^s \Big\{ \frac{(\lambda-\nu)[e^{\nu x}\cosh_+(\mu x) + e^{\lambda x}\cosh_-(\mu x)]}{2[(\lambda-\nu)^2-4\mu^2]} + + \frac{\mu[e^{\nu x}\sinh_+(\mu x) - e^{\lambda x}\sinh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2} \Big\}_{\nu=\lambda}. \tag{6}$$

In particular

$$e_-^{(\lambda+\mu)x} \boxast e_+^{(\lambda-\mu)x} + e_-^{(\lambda-\mu)x} \boxast e_+^{(\lambda+\mu)x} = e^{\lambda x}[\sinh_-(\mu x) - \sinh_+(\mu x)].$$

The following equations follow similarly.

$$[x^r e^{\lambda_1 x}\cosh_-(\lambda_2 x)] \boxast [x^s e^{\mu_1 x}\sinh_+(\mu_2 x)] = = D_{\lambda_1}^r D_{\mu_1}^s \Big\{ \frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\sinh_+(\mu_2 x) + e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]} + - \frac{(\lambda_2-\mu_2)[e^{\mu_1 x}\cosh_+(\mu_2 x) + e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]} +$$

$$+\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\sinh_+(\mu_2 x)+e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}+$$

$$+\frac{(\lambda_2+\mu_2)[e^{\mu_1 x}\cosh_+(\mu_2 x)-e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}\Big\}, \tag{7}$$

$$[x^r e^{\lambda_1 x}\sinh_-(\lambda_2 x)] \boxed{*} [x^s e^{\mu_1 x}\cosh_+(\mu_2 x)] =$$

$$= D^r_{\lambda_1} D^s_{\mu_1}\Big\{\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\sinh_+(\mu_2 x)+e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+$$

$$-\frac{(\lambda_2-\mu_2)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+$$

$$-\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\sinh_+(\mu_2 x)-e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}+$$

$$-\frac{(\lambda_2+\mu_2)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}\Big\}, \tag{8}$$

$$[x^r e^{\lambda_1 x}\sinh_-(\lambda_2 x)] \boxed{*} [x^s e^{\mu_1 x}\sinh_+(\mu_2 x)] =$$

$$= D^r_{\lambda_1} D^s_{\mu_1}\Big\{\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+$$

$$-\frac{(\lambda_2-\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)+e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2-\mu_2)^2]}+$$

$$-\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cosh_+(\mu_2 x)+e^{\lambda_1 x}\cosh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}+$$

$$-\frac{(\lambda_2+\mu_2)[e^{\mu_1 x}\sinh_+(\mu_2 x)-e^{\lambda_1 x}\sinh_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2-(\lambda_2+\mu_2)^2]}\Big\}, \tag{9}$$

if $(\lambda_1-\mu_1)^2 \neq (\lambda_2 \pm \mu_2)^2$ and

$$[x^r e^{\lambda x}\cosh_-(\mu x)] \boxed{*} [x^s e^{\lambda x}\sinh_+(\mu x)] =$$

$$= \tfrac{1}{2}B(r+1,s+1)\,\mathrm{sgn}\,x.x^{r+s+1}e^{\lambda x}\sinh(\mu x)+$$

$$+D^r_\lambda D^s_\nu\Big\{\frac{(\lambda-\nu)[e^{\nu x}\sinh_+(\mu x)-e^{\lambda x}\sinh_-(\mu x)]}{2[(\lambda-\nu)^2-4\mu^2]}+$$

$$+\frac{\mu[e^{\nu x}\cosh_+(\mu x)+e^{\lambda x}\cosh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2}\Big\}_{\nu=\lambda}, \tag{10}$$

$$[x^r e^{\lambda x}\sinh_-(\mu x)] \boxed{*} [x^s e^{\lambda x}\cosh_+(\mu x)] =$$

$$= \tfrac{1}{2}\,B(r+1,s+1)\,\mathrm{sgn}\,x.x^{r+s+1}e^{\lambda x}\sinh(\mu x) +$$
$$-D_\lambda^r D_\nu^s\Big\{\frac{(\lambda-\nu)[e^{\nu x}\sinh_+(\mu x) - e^{\lambda x}\sinh_-(\mu x)]}{2[(\lambda-\nu)^2-4\mu^2]} +$$
$$+\frac{\mu[e^{\nu x}\cosh_+(\mu x) + e^{\lambda x}\cosh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2}\Big\}_{\nu=\lambda}, \tag{11}$$

$$[x^r e^{\lambda x}\sinh_-(\mu x)] \boxed{*} [x^s e^{\lambda x}\sinh_+(\mu x)] =$$
$$= \tfrac{1}{2}\,B(r+1,s+1)\,\mathrm{sgn}\,x.x^{r+s+1}e^{\lambda x}\cosh(\mu x) +$$
$$-D_\lambda^r D_\nu^s\Big\{\frac{(\lambda-\nu)[e^{\nu x}\cosh_+(\mu x) + e^{\lambda x}\cosh_-(\mu x)]}{2[(\lambda-\nu)^2-4\mu^2]} +$$
$$+\frac{\mu[e^{\nu x}\sinh_+(\mu x) - e^{\lambda x}\sinh_-(\mu x)]}{(\lambda-\nu)^2-4\mu^2}\Big\}_{\nu=\lambda}, \tag{12}$$

if $\mu \neq 0$.

We now define the locally summable functions $\cos_+(\lambda x)$, $\cos_-(\lambda x)$, $\sin_+(\lambda x)$ and $\sin_-(\lambda x)$ by

$$\cos_+(\lambda x) = \begin{cases} \cos(\lambda x), & x>0, \\ 0, & x<0, \end{cases} \qquad \cos_-(\lambda x) = \begin{cases} 0, & x>0, \\ \cos(\lambda x), & x<0, \end{cases}$$

$$\sin_+(\lambda x) = \begin{cases} \sin(\lambda x), & x>0, \\ 0, & x<0, \end{cases} \qquad \sin_-(\lambda x) = \begin{cases} 0, & x>0, \\ \sin(\lambda x), & x<0. \end{cases}$$

It follows that

$$\cos_+(\lambda x) = \frac{e_+^{i\lambda x} + e_+^{-i\lambda x}}{2}, \qquad \cos_-(\lambda x) = \frac{e_-^{i\lambda x} + e_-^{-i\lambda x}}{2},$$
$$\sin_+(\lambda x) = \frac{e_+^{i\lambda x} - e_+^{-i\lambda x}}{2i}, \qquad \sin_-(\lambda x) = \frac{e_-^{i\lambda x} - e_-^{-i\lambda x}}{2i},$$
$$\cos_-(\lambda x) + \cos_+(\lambda x) = \cos(\lambda x), \qquad \sin_-(\lambda x) + \sin_+(\lambda x) = \sin(\lambda x).$$

We now note that if we replace λ_2 by $i\lambda_2$ and μ_2 by $i\mu_2$ in equations (5), (7), (8) and (9) we get expressions for neutrix convolution products of the form

$$[x^r e^{\lambda_1 x}\cos_-(\lambda_2 x)] \boxed{*} [x^s e^{\mu_1 x}\cos_+(\mu_2 x)]$$

and if we replace μ by $i\mu$ in equations (6), (10), (11) and (12) we get expressions for neutrix convolution products of the form

$$[x^r e^{\lambda x}\cos_-(\mu x)] \boxed{*} [x^s e^{\lambda x}\cos_+(\mu x)].$$

For example,

$$[x^r e^{\lambda_1 x}\cos_-(\lambda_2 x)] \boxed{*} [x^s e^{\mu_1 x}\cos_+(\mu_2 x)] =$$
$$= D^r_{\lambda_1} D^s_{\mu_1}\Big\{\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cos_+(\mu_2 x)+e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2+(\lambda_2-\mu_2)^2]}+$$
$$+\frac{(\lambda_2-\mu_2)[e^{\mu_1 x}\sin_+(\mu_2 x)+e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2+(\lambda_2-\mu_2)^2]}+$$
$$+\frac{(\lambda_1-\mu_1)[e^{\mu_1 x}\cos_+(\mu_2 x)+e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2+(\lambda_2+\mu_2)^2]}+$$
$$-\frac{(\lambda_2+\mu_2)[e^{\mu_1 x}\sin_+(\mu_2 x)-e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{2[(\lambda_1-\mu_1)^2+(\lambda_2+\mu_2)^2]}\Big\}.$$

Finally, we finish with a list of further commutative neutrix convolution products which have been obtained earlier in other journals.

$$\cos_-(\lambda x) \boxed{*} \cos_+(\mu x) = \frac{\lambda\sin_-(\lambda x)+\mu\sin_+(\mu x)}{\lambda^2-\mu^2},$$
$$\cos_-(\lambda x) \boxed{*} \sin_+(\mu x) = -\frac{\mu\cos_-(\lambda x)+\mu\cos_+(\mu x)}{\lambda^2-\mu^2},$$
$$\sin_-(\lambda x) \boxed{*} \cos_+(\mu x) = -\frac{\lambda\cos_-(\lambda x)+\lambda\cos_+(\mu x)}{\lambda^2-\mu^2},$$
$$\sin_-(\lambda x) \boxed{*} \sin_+(\mu x) = -\frac{\mu\sin_-(\lambda x)+\lambda\sin_+(\mu x)}{\lambda^2-\mu^2},$$

for $\lambda \neq \pm\mu$,

$$\cos_-(\lambda x) \boxed{*} \cos_+(\lambda x) = \frac{2\lambda x[\cos_-(\lambda x)-\cos_+(\lambda x)]+\sin_-(\lambda x)-\sin_+(\lambda x)}{4\lambda},$$
$$\cos_-(\lambda x) \boxed{*} \sin_+(\lambda x) = \frac{2\lambda x[\sin_-(\lambda x)-\sin_+(\lambda x)]+\cos(\lambda x)}{4\lambda},$$
$$\sin_-(\lambda x) \boxed{*} \cos_+(\lambda x) = -\frac{2\lambda x[\sin_+(\lambda x)-\sin_-(\lambda x)]+\cos(\lambda x)}{4\lambda},$$
$$\sin_-(\lambda x) \boxed{*} \sin_+(\lambda x) = \frac{2\lambda x[\cos_+(\lambda x)-\cos_-(\lambda x)]+\sin_-(\lambda x)-\sin_+(\lambda x)}{4\lambda},$$

for $\lambda \neq 0$, see [3],

$$x_-^\lambda \boxed{*} x_+^\mu = B(-\lambda-\mu-1,\mu+1)x^{\lambda+\mu+1},$$

for $\lambda, \mu, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots$,

$$\begin{aligned} x_-^\lambda \boxed{*}\, x_+^s &= (-1)^{s+1} B(\lambda+1, s+1) x_-^{\lambda+s+1}, \\ x_+^\lambda \boxed{*}\, x_-^s &= (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1} \end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$,

$$x_-^r \boxed{*}\, x_+^s = -B(r+1, s+1)[(-1)^r x_+^{r+s+1} + (-1)^s x_-^{r+s+1}]$$

for $r, s = 0, 1, 2, \ldots$, see [4],

$$\begin{aligned} x_-^\lambda \boxed{*}\, x_+^{r-\lambda} &= B(-r-1, r+1-\lambda) x_-^{r+1} + B(-r-1, \lambda+1) x_+^{r+1} + \\ &\quad + \frac{(-1)^r (\lambda)_{r+1}}{(r+1)!} x^{r+1} \ln|x| \end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = -1, 0, 1, 2, \ldots$,

$$x_-^\lambda \boxed{*}\, x_+^{-r-\lambda} = \frac{\pi \cot(\pi\lambda)}{(-1-\lambda)_{r-1}} \delta^{(r-2)}(x) - \frac{(-1)^r (r-2)!}{(-1-\lambda)_{r-1}} x^{-r+1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 2, 3, \ldots$,

$$x_-^{s-1/2} \boxed{*}\, x_+^{-r-s+1/2} = -\frac{(-1)^r (r-2)!}{(-\frac{1}{2} - s)_{r-1}} x^{-r+1}$$

for $s = 0, 1, 2, \ldots$ and $r = 2, 3, \ldots$, see [5],

$$\ln x_- \boxed{*} \ln x_+ = -(\pi^2/6 + 1)|x| + |x| \ln|x| - \tfrac{1}{2} |x| \ln^2|x|,$$

see [7] and

$$\ln x_- \boxed{*}\, x_+^r = \frac{(-1)^{r+1}}{r+1} x_-^{r+1} \ln x_- + \frac{(-1)^r \phi(r+1)}{r+1} x_-^{r+1} + \frac{\phi(r)}{r+1} x_+^{r+1}$$

$r = 0, 1, 2, \ldots$, see [6], where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r = 1, 2, \ldots, \\ 0, & r = 0. \end{cases}$$

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Received by the editors November 18, 1996.