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ON THE TRIPLE g-INTEGRAL

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Abstract

In the framework of the Pap g-calculus the notion of double g-integral has been introduced. In [2] A. Marková and B. Riečan has presented the Fubini theorem, the integral transformation formula and the Green theorem. We will extend their means to the triple g-integral, the Gauss-Ostrogradsky theorem and Stokes theorem.

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0. Introduction

The g-calculus of E. Pap ([4], see also [5]), is based on a function $g : [a, b] \to [0, \infty]$ which is stricly monotone, bijective and such that either $g(a) = 0$, or $g(b) = 0$. The g-integral of a measurable function $f : [\alpha, \beta] \to [a, b]$ is defined by the formula

$$\int_{[\alpha,\beta]}^{\oplus} f(x)dx = g^{-1}\left(\int_{\alpha}^{\beta} g(f(x))dx\right).$$

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In [2] A. Markova and B. Riecan is defined the double g-integral by the formula

$$\iint\limits_D^{\oplus} f(x,y)dxdy = g^{-1}\left(\iint\limits_D g(f(x,y))dxdy\right).$$

and the curvilinear g-integral

$$\int\limits_C^{\oplus} f(x,y)dx = g^{-1}\left(\int\limits_C g(f(x,y))dx\right).$$

Since the curvilinear integral does not need to be non-negative, although g is non-negative, they extend the range of the function g to $[-\infty,\infty]$ (for more information see [2], [1] and also [3]). In addition to the curvilinear integral we will also use the surface integral therefore, and we also have to work with a generator g specified as follows: $g:[a,b]\to[-\infty,\infty]$.

1. Preliminaries

Let be given a generator $g:[a,b]\to[-\infty,\infty]$, which is a bijective mapping, decreasing or increasing. We define on the interval $[a,b]$ three binary operations :

the pseudo-addition by the formula

$$u\oplus v = g^{-1}\left(g(u)+g(v)\right),$$

the pseudo-multiplication by the formula

$$u\otimes v = g^{-1}\left(g(u).g(v)\right),$$

and the pseudo-difference by the formula

$$u\ominus v = g^{-1}\left(g(u)-g(v)\right).$$

The associativity of a given pseudo-addition and a pseudo-multiplication allows us to extend them (using induction) to n-ary ($n\in N$) operation in the following way :

$$^{\oplus}\sum_{k=1}^{n} u_k = (u_1 \oplus u_2 \oplus \ldots \oplus u_n) = g^{-1}\left(\sum_{k=1}^{n} g(u_k)\right)$$

$$^{\otimes}\prod_{k=1}^{n} u_k = (u_1 \otimes u_2 \otimes \ldots \otimes u_n) = g^{-1}\left(\prod_{k=1}^{n} g(u_k)\right).$$

2. Properties of the integral

Proposition 1. *Let D be a measurable set, $f_1, f_2 : D \to [a,b]$ be g-integrable functions on D, $\alpha \in [a,b]$, D_1, D_2 be measurable, nonoverlapping sets with $D_1 \cup D_2 = D$. Then*

$$\iiint_D^{\oplus} f_1(x,y,z)dxdydz \oplus \iiint_D^{\oplus} f_2(x,y,z)dxdydz = \iiint_D^{\oplus} [f_1(x,y,z) \oplus f_2(x,y,z)]\, dxdydz$$

$$\alpha \otimes \iiint_D^{\oplus} f_1(x,y,z)dxdydz = \iiint_D^{\oplus} [\alpha \otimes f_1(x,y,z)]\, dxdydz$$

$$\iiint_D^{\oplus} f_1(x,y,z)dxdydz = \iiint_{D_1}^{\oplus} f_1(x,y,z)dxdydz \oplus \iiint_{D_2}^{\oplus} f_1(x,y,z)dxdydz.$$

Proof. The proof is analogous to the proof of the corresponding proposition in [2] □

Theorem 1. *Let D be an elementary region defined by the equality:*
$D = \{(x,y,z) \in R^3;\ c \le x \le d;\ \phi(x) \le y \le \psi(x);\ r(x,y) \le z \le s(x,y)\}$, *where ϕ, ψ be continuous on $[c,d]$, $\phi(x) \le \psi(x)$ for all $x \in [c,d]$ and r, s be continuous on $D^* = \{(x,y) \in R^2;\ c \le x \le d;\ \phi(x) \le y \le \psi(x)\}$, $r(x,y) \le s(x,y)$ for all $(x,y) \in D^*$. Let f be continuous on D. Then f is g-integrable on D and*

$$\iiint_D^{\oplus} f(x,y,z)dxdydz = \int_{[c,d]}^{\oplus} \left[\int_{[\phi(x),\psi(x)]}^{\oplus} \left(\int_{[r(x,y),s(x,y)]}^{\oplus} f(x,y,z)dz \right) dy \right] dx.$$

Proof. The proof is analogous to the proof of the corresponding proposition in [2] □.

3. Gauss-Ostrogradsky theorem

Let S be a piecewise smooth simple oriented surface, L, M, N be real functions such that their domains contain S, $L, M, N : S \to [a, b]$. Then we define

$$\overset{\oplus}{\iint_S} L(x,y,z)dydz = g^{-1}\left(\iint_S g(L(x,y,z))dydz\right)$$
$$\overset{\oplus}{\iint_S} M(x,y,z)dzdx = g^{-1}\left(\iint_S g(M(x,y,z))\,dzdx\right)$$
$$\overset{\oplus}{\iint_S} N(x,y,z)dxdy = g^{-1}\left(\iint_S g(N(x,y,z))dxdy\right).$$

Proposition 2. *If* $g(L(x,y,z))\mathbf{i} + g(M(x,y,z))\mathbf{j} + g(N(x,y,z))\mathbf{k}$ *is integrable on a piecewise smooth simply oriented surface* S, *then*

$$\overset{\oplus}{\iint_S} L(x,y,z)dydz \oplus \overset{\oplus}{\iint_S} M(x,y,z)dzdx \oplus \overset{\oplus}{\iint_S} N(x,y,z)dxdy =$$
$$= g^{-1}\left(\iint_S g(L(x,y,z))dydz + g(M(x,y,z))dzdx + g(N(x,y,z))dxdy\right).$$

Proof.

$$\overset{\oplus}{\iint_S} L(x,y,z)dydz \oplus \overset{\oplus}{\iint_S} M(x,y,z)dzdx \oplus \overset{\oplus}{\iint_S} N(x,y,z)dxdy =$$
$$= g^{-1}\left[g\left(\overset{\oplus}{\iint_S} Ldydz\right) + g\left(\overset{\oplus}{\iint_S} Mdzdx\right) + g\left(\overset{\oplus}{\iint_S} Ndxdy\right)\right] =$$

$$= g^{-1}\left[g\left(g^{-1}\left(\iint_S g(L)dydz\right)\right) + g\left(g^{-1}\left(\iint_S g(M)dzdx\right)\right) + \right.$$

$$\left. + g\left(g^{-1}\left(\iint_S g(N)dxdy\right)\right)\right] =$$

$$= g^{-1}\left(\iint_S g(L)dydz + \iint_S g(M)dzdx + \iint_S g(N)dxdy\right) =$$

$$= g^{-1}\left(\iint_S g(L(x,y,z))dydz + g(M(x,y,z))dzdx + g(N(x,y,z))dxdy\right).$$

□

In the next Proposition we will prove the formula which allows us to compute the surface g-integral with the aid of the double g-integral. Let S be a surface given by the equations in parametric form, i.e., $F(u,v) = [\phi(u,v), \psi(u,v), \chi(u,v)]$. We will deal with a piecewise smooth surface, i.e., partial derivates of functions ϕ, ψ, χ with respect to all coordinates are continuous and

$$|F_u \times F_v| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \phi_u & \psi_u & \chi_u \\ \phi_v & \psi_v & \chi_v \end{vmatrix} \neq 0$$

In our particular case we should assume

$$\begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix} \neq 0 \tag{1}$$

Proposition 3. *Let S be a piecewise smooth simply oriented surface, given by the function $P = F(u,v)$, $(u,v) \in B$ ($B \subset R^2$ is a closed bounded simply connected region whose boundary ∂B is piecewise smooth). Let $F(u,v) = [\phi(u,v), \psi(u,v), \chi(u,v)]$, i.e. the surface S is given by the equations in parametric form:*

$$x = \phi(u,v), \; y = \psi(u,v), \; z = \chi(u,v), \qquad (u,v) \in B$$

which satisfy (1) for all $(u,v) \in B$ and N be a real function continuous on S, $N : S \to [a,b]$. Then

$$\iint_S^{\oplus} N(x,y,z)dxdy = \iint_B^{\oplus} N(\phi,\psi,\chi) \otimes g^{-1}\left(\begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}\right) dudv.$$

Proof.

$$\iint\limits_S^{\oplus} N(x,y,z)dxdy = g^{-1}\left(\iint\limits_S g(N(x,y,z))dxdy\right) =$$
$$= g^{-1}\left(\iint\limits_B g(N(\phi,\psi,\chi)) \cdot \begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix} dudv\right) =$$
$$= g^{-1}\left(\iint\limits_B g\left[g^{-1}\left(g(N(\phi,\psi,\chi)) \cdot g\left(g^{-1}\left(\begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}\right)\right)\right)\right] dudv\right) =$$
$$= g^{-1}\left(\iint\limits_B g\left[N(\phi,\psi,\chi)) \otimes g^{-1}\left(\begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}\right)\right] dudv\right) =$$
$$= \iint\limits_B^{\oplus} N(\phi,\psi,\chi) \otimes g^{-1}\left(\begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix}\right) dudv. \quad \square$$

Theorem 2. *Let S be a simply closed piecewise smooth surface, which is oriented in the direction of outer normal. Let D is the set, which contains points of the surface S and interior points of the surface S. Let g be differentiable on (a,b), $L, M, N : D \to [a,b]$ have continuous partial derivatives on D. Then*

$$\iint\limits_S^{\oplus} L(x,y,z)dydz \oplus \iint\limits_S^{\oplus} M(x,y,z)dzdx \oplus \iint\limits_S^{\oplus} N(x,y,z)dxdy =$$
$$= \iiint\limits_D \left(\frac{\partial^{\oplus} L(x,y,z)}{\partial x} \oplus \frac{\partial^{\oplus} M(x,y,z)}{\partial y} \oplus \frac{\partial^{\oplus} N(x,y,z)}{\partial y}\right) dxdydz.$$

Proof. By the definition:

$$\frac{\partial^{\oplus} L(x,y,z)}{\partial x} = g^{-1}\left(\frac{\partial g(L(x,y,z))}{\partial x}\right)$$
$$\frac{\partial^{\oplus} M(x,y,z)}{\partial y} = g^{-1}\left(\frac{\partial g(M(x,y,z))}{\partial y}\right)$$
$$\frac{\partial^{\oplus} N(x,y,z)}{\partial z} = g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial z}\right)$$

hence

$$\frac{\partial^{\oplus} L(x,y,z)}{\partial x} \oplus \frac{\partial^{\oplus} M(x,y,z)}{\partial y} \oplus \frac{\partial^{\oplus} N(x,y,z)}{\partial z} =$$
$$= g^{-1}\left(\frac{\partial g(L(x,y,z))}{\partial x}\right) \oplus g^{-1}\left(\frac{\partial g(M(x,y,z))}{\partial y}\right) \oplus g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial z}\right) =$$
$$= g^{-1}\left(g\left[g^{-1}\left(\frac{\partial g(L)}{\partial x}\right)\right] + g\left[g^{-1}\left(\frac{\partial g(M)}{\partial y}\right)\right] + g\left[g^{-1}\left(\frac{\partial g(N)}{\partial z}\right)\right]\right) =$$
$$= g^{-1}\left(\frac{\partial g(L(x,y,z))}{\partial x} + \frac{\partial g(M(x,y,z))}{\partial y} + \frac{\partial g(N(x,y,z))}{\partial z}\right).$$

Therefore

$$\iiint_D \left(\frac{\partial^{\oplus} L(x,y,z)}{\partial x} \oplus \frac{\partial^{\oplus} M(x,y,z)}{\partial y} \oplus \frac{\partial^{\oplus} N(x,y,z)}{\partial y}\right) dxdydz =$$
$$= g^{-1}\left(\iiint_D g\left[g^{-1}\left(\frac{\partial^{\oplus} g(L)}{\partial x} \oplus \frac{\partial^{\oplus} g(M)}{\partial y} \oplus \frac{\partial^{\oplus} g(N)}{\partial y}\right)\right] dxdydz\right) =$$
$$= g^{-1}\left(\iiint_D \left(\frac{\partial^{\oplus} g(L)}{\partial x} \oplus \frac{\partial^{\oplus} g(M)}{\partial y} \oplus \frac{\partial^{\oplus} g(N)}{\partial y}\right) dxdydz\right). \qquad (*)$$

On the other hand, by Proposition 2 and the Gauss-Ostrogradsky theorem we obtain

$$\overset{\oplus}{\iint_S} L(x,y,z)dydz \oplus \overset{\oplus}{\iint_S} M(x,y,z)dzdx \oplus \overset{\oplus}{\iint_S} N(x,y,z)dxdy =$$
$$= g^{-1}\left(\iint_S g(L)dydz + \iint_S g(M)dzdx + \iint_S g(N)dxdy\right) =$$
$$= g^{-1}\left(\iiint_D \left(\frac{\partial^{\oplus} g(L)}{\partial x} \oplus \frac{\partial^{\oplus} g(M)}{\partial y} \oplus \frac{\partial^{\oplus} g(N)}{\partial y}\right) dxdydz\right).$$

By the equality and (*) the stated formula is proved. □

4. Stokes theorem

Let $C \subset R^2$ be a regular oriented curve, L, M, N be a real functions such that their domains contain C, $L, M, N : C \to [a, b]$. Then we define

$$\int_C^{\oplus} L(x,y,z)dx = g^{-1}\left(\int_C g(L(x,y,z))dx\right)$$
$$\int_C^{\oplus} M(x,y,z)dy = g^{-1}\left(\int_C g(M(x,y,z))dy\right)$$
$$\int_C^{\oplus} N(x,y,z)dz = g^{-1}\left(\int_C g(N(x,y,z))dz\right).$$

Proposition 4. *If* $g(L(x,y,z))\mathbf{i} + g(M(x,y,z))\mathbf{j} + g(N(x,y,z))\mathbf{k}$ *is integrable on a regular oriented curve* C*, then*

$$\int_C^{\oplus} L(x,y,z)dx \oplus \int_C^{\oplus} M(x,y,z)dy \oplus \int_C^{\oplus} N(x,y,z)dz =$$
$$= g^{-1}\left(\int_C g(L(x,y,z))dx + g(M(x,y,z))dy + g(N(x,y,z))dz\right).$$

Proof. We have

$$\int_C^{\oplus} L(x,y,z)dx \oplus \int_C^{\oplus} M(x,y,z)dy \oplus \int_C^{\oplus} N(x,y,z)dz =$$
$$= g^{-1}\left[g\left(\int_C^{\oplus} Ldx\right) + g\left(\int_C^{\oplus} Mdy\right) + g\left(\int_C^{\oplus} Ndz\right)\right] =$$
$$= g^{-1}\left[g\left(g^{-1}\left(\int_C g(L)dx\right)\right) + g\left(g^{-1}\left(\int_C g(M)dy\right)\right) + \right.$$
$$\left. + g\left(g^{-1}\left(\int_C g(N)dz\right)\right)\right] =$$
$$= g^{-1}\left(\int_C g(L(x,y,z))dx + g(M(x,y,z))dy + g(N(x,y,z))dz\right). \quad \square$$

Theorem 3. *Let S be an orientable surface with boundary C, where the equation defining each simple smooth part of S have two continuous derivatives. Let g be differentiable on (a,b) and L, M, N have continuous partial derivatives on S. Then*

$$\int_C^{\oplus} L(x,y,z)dx \oplus \int_C^{\oplus} M(x,y,z)dy \oplus \int_C^{\oplus} N(x,y,z)dz =$$
$$= \iint_S^{\oplus} \left(\frac{\partial^{\oplus} N}{\partial y} \ominus \frac{\partial^{\oplus} M}{\partial z}\right) dydz \oplus \iint_S^{\oplus} \left(\frac{\partial^{\oplus} L}{\partial z} \ominus \frac{\partial^{\oplus} N}{\partial x}\right) dzdx \oplus$$
$$\oplus \iint_S^{\oplus} \left(\frac{\partial^{\oplus} M}{\partial x} \ominus \frac{\partial^{\oplus} L}{\partial y}\right) dxdy.$$

Proof. By the definition

$$\frac{\partial^{\oplus} N(x,y,z)}{\partial y} \ominus \frac{\partial^{\oplus} M(x,y,z)}{\partial z} =$$
$$= g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial y}\right) \ominus g^{-1}\left(\frac{\partial g(M(x,y,z))}{\partial z}\right) =$$
$$= g^{-1}\left\{g\left[g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial y}\right)\right] - g\left[g^{-1}\left(\frac{\partial g(M(x,y,z))}{\partial z}\right)\right]\right\} =$$
$$= g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial y} - \frac{\partial g(M(x,y,z))}{\partial z}\right).$$

Then

$$\iint_S^{\oplus} \left(\frac{\partial^{\oplus} g(N(x,y,z))}{\partial y} \ominus \frac{\partial^{\oplus} g((M(x,y,z))}{\partial z}\right) dydz =$$
$$= g^{-1}\left\{\iint_S g\left[g^{-1}\left(\frac{\partial g(N(x,y,z))}{\partial y} - \frac{\partial g(M(x,y,z))}{\partial z}\right)\right] dydz\right\} =$$
$$= g^{-1}\left[\iint_S \left(\frac{\partial g(N(x,y,z))}{\partial y} - \frac{\partial g(M(x,y,z))}{\partial z}\right) dydz\right]$$

Therefore we can write the right-hand side of the equality, given in the theorem, in the following way:

$$g^{-1}\left[\iint\limits_S\left(\frac{\partial g(N)}{\partial y}-\frac{\partial g(M)}{\partial z}\right)dydz+\iint\limits_S\left(\frac{\partial g(L)}{\partial z}-\frac{\partial g(N)}{\partial x}\right)dzdx+\right.$$

$$\left.+\iint\limits_S\left(\frac{\partial g(M)}{\partial x}-\frac{\partial g(L)}{\partial y}\right)dxdy\right].$$

Hence by the Stokes theorem it equals

$$g^{-1}\left(\int\limits_C g(L(x,y,z))dx+g(M(x,y,z))dy+g(N(x,y,z))dz\right).$$

If we use the Proposition 4 for the left-hand side of the equality from theorem, the formula written in the theorem is proved. □

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