NOVI SAD J. MATH.
VOL. 27, NO. 2, 1997, 37-47

THE CONTINUITY OF THE NULL-ADDITIVE FUZZY MEASURES

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We investigate the connections between different types of continuity of the null-additive fuzzy measure m (monotone set function which vanishes at the empty set and such that $m(B) = 0$ implies $m(A \cup B) = m(A)$) as continuity from above, continuity from below, autocontinuity from above, autocontinuity from below, continuity with respect to other fuzzy measure. A Lebesgue type decomposition theorem for continuous and autocontinuous from above fuzzy measure is proved.

AMS Mathematics Subject Classification (1991): 28A10, 28E10
Key words and phrases: fuzzy measure, continuous from above, continuous from below, autocontinuous from above, autocontinuous from below

1. Introduction

Fuzzy measures were introduced and investigated by Sugeno [22]. Monotone set functions which vanish at the empty set were investigated before Sugeno by G.Choquet [3] under the name capacity. These set functions have many important applications in different fields, for example [1], [2], [8], [19], [21], [22], [27].

Definition 1. *A fuzzy measure $m, m : \Sigma \to [0,\infty]$, is a non-negative extended real-valued set function m defined on σ-ring Σ and with the properties:*

(FM_1) $$m(\emptyset) = 0,$$

(FM_2) $$E \subset F \quad \Rightarrow \quad m(E) \leq m(F).$$

In some papers ([23],[25],[26]) fuzzy measures have two continuity properties more:

(FM_3) $$E_1 \subset E_2 \subset \ldots \quad , E_n \in \Sigma \quad \Rightarrow \quad m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n),$$

(FM_4) $E_1 \supset E_2 \supset \ldots$ $, E_n \in \Sigma$ and there exists n_0 such that $m(E_{n_0}) < \infty \Rightarrow \mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

In this paper we shall call the properties (FM_3) and (FM_4) of m the continuity from below and the continuity from above, respectively.
We shall investigate the connections between different types of continuities of special fuzzy measures, the so called null-additive ([18], [20], [24], [25], [26]) fuzzy measures.

2. Continuity

Throughout this paper Σ always denotes a σ-ring of subsets of the given set X.

According to Wang [25] we have

Definition 2. *A set function m, $m : \Sigma \to [0,\infty]$, is called null-additive, if we have*
$$m(A \cup B) = m(A)$$
whenever $A, B \in \Sigma$, $A \cap B = \emptyset$, and $m(B) = 0$.

Example 1. A generalization of $\perp$-decomposable measure ([14]-[18],[22],[27]) is the $\bigoplus$ - decomposable measure $m : \Sigma \to [a,b]$ ([9],[11],[19]). Let $[a,b]$ be a closed (in some cases semiclosed, see [19]) real interval. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative, and has a zero element denoted by $\mathbf{0}$, which is either a or b. A set function $m : \Sigma \to [a,b]$ is a $\bigoplus$-decomposable measure if there hold $m(\emptyset) = \mathbf{0}$ and

$$m(A \cup B) = m(A) \oplus m(B)$$

whenever $A, B \in \Sigma$ *and* $A \cap B = \emptyset$. Then, m is null-additive.

There are null-additive set functions which are not fuzzy measures.

Example 2. A set function $m : \Sigma \to [0, \infty)$ is said to be k-triangular for $k \geq 1$ if $m(\emptyset) = 0$ and

$$m(A) - km(B) \leq m(A \cup B) \leq m(A) + km(B),$$

whenever $A, B \in \Sigma, A \cap B = \emptyset$ ([6], [7], [12], [13]).
More generally, for a uniform semigroup S (with a neutral element $\mathbf{0}$) valued set function $\mu : \Sigma \to S$ is said to be k-triangular if the preceding inequality for $m_i(A) = f_i(\mu(A))$ $(i \in I)$ holds, where $\{f_i\}_{i \in I}$ is the family of triangular functionals, i.e. such that

$$| f_i(x) - f_i(y) | \leq f_i(x + y) \leq f_i(x) + f_i(y) \quad (x, y \in S) \quad \text{and} \quad f_i(\mathbf{0}) = 0,$$

defined by

$$d_i(x, \mathbf{0}) = f_i(x) \quad (x \in S),$$

where $\{d_i\}_{i \in I}$ is the family of pseudometrics which generates the uniformity in S ([12], [13]). It is obvious that a k-triangular set function m is always null-additive, although it may not be monotone.

For other examples see [18].

Definition 3. *A set function m is called autocontinuous from above (resp. from below) if for every $\epsilon > 0$ and every $A \in \Sigma$, there exists $\delta = \delta(A, \epsilon) > 0$ such that*

$$m(A) - \epsilon \leq m(A \cup B) \leq m(A) + \epsilon \ (\textit{resp. } m(A) - \epsilon \leq m(A \setminus B) \leq m(A) + \epsilon)$$

whenever $B \in \Sigma, A \cap B = \emptyset$ (resp. $B \subset A$) and $m(B) < \delta$ holds.

By Proposition 3 from [25] any set function which is autocontinuous from above (below) is null-additive. Wang proved in Theorem 1 from [26] that for a finite fuzzy measure which is continuous from above and continuous from below, the autocontinuity from above is equivalent to the autocontinuity from below. We examine now this relationship in more detail.

Theorem 1. *If m is a finite fuzzy measure which is autocontinuous from below and continuous from above, then it is autocontinuous from above.*

Proof. Suppose that the theorem is not true. Then there exist $\epsilon > 0$, a set E from Σ and a sequence $\{F_n\}$ from Σ such that

$$m(F_n) < \frac{1}{n} \quad and \quad m(E \cup F_n) > m(E) + \epsilon \quad (n \in N).$$

Since m is autocontinuous from below there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that

$$(1) \qquad m(E \cup (F_{n_s} \setminus \bigcup_{i=s+1}^{k+1} F_{n_i}) > m(E) + \epsilon$$

for $s = 1, 2, ..., k$. Since m is a fuzzy measure we have

$$(2) \qquad m(E \cup (\bigcup_{i=s}^{\infty}(F_{n_s} \setminus \bigcup_{i=s+1}^{\infty} F_{n_i}))) \geq m(E \cup (F_{n_s} \setminus \bigcup_{i=s+1}^{\infty} F_{n_i})).$$

By the continuity from above of m and (1) it follows

$$m(E \cup (F_{n_s} \setminus \bigcup_{i=s+1}^{\infty} F_{n_i})) > m(E) + \epsilon.$$

Hence by (2) we obtain

$$m(E \cup (\bigcup_{i=s}^{\infty}(F_{n_s} \setminus \bigcup_{i=s+1}^{\infty} F_{n_i}))) > m(E) + \epsilon.$$

Since m is continuous from above, taking $s \to \infty$ in the last inequality we obtain the contradiction

$$m(E) \geq m(E) + \epsilon.$$

Theorem 2. *If m is a finite fuzzy measure which is autocontinuous from above and continuous from below, then it is autocontinuous from below.*

Proof. Suppose that the theorem is not true. Then there exist $\epsilon > 0, E$ from Σ and a sequence $\{F_n\}$ from Σ such that

(3) $$m(F_n) < \frac{1}{n} \quad and \quad m(E) > m(E \setminus F_n) + \epsilon \quad (n \in N).$$

Since m is autocontinuous from above there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that

$$m(\bigcup_{i=s}^{k} F_{n_i}) < \frac{1}{s} \quad for \quad s = 1, 2, ..., k.$$

The continuity from below implies

(4) $$m(\bigcup_{i=s}^{\infty} F_{n_i}) \leq \frac{1}{s} \quad (s \in N).$$

Since m is a fuzzy measure we have

$$m(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} F_{n_i}) \leq m(\bigcup_{i=s}^{\infty} F_{n_i}).$$

Hence by (4)

(5) $$m(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} F_{n_i}) = 0.$$

By the continuity from below of m we have

$$\lim_{s \to \infty} m(E \setminus \bigcup_{i=s}^{\infty} F_{n_i}) = m(E \setminus \bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} F_{n_i}).$$

Hence by, the null-additivity of m and (5) we obtain (Proposition 2 (4),[25])

(6) $$\lim_{s \to \infty} m(E \setminus \bigcup_{i=s}^{\infty} F_{n_i}) = m(E).$$

Since m is a fuzzy measure we have

$$m(E \setminus F_{n_i}) \geq m(E \setminus \bigcup_{i=s}^{\infty} F_{n_i}).$$

Hence by (3) and (6)

$$m(E) - \epsilon > m(E).$$

Contradiction.

Theorems 1 and 2 imply

Corollary 1. *(Theorem 1,[26]) If m is a finite fuzzy measure which is continuous from above and continuous from below, then the autocontinuity from above and the autocontinuity from below are equivalent for m.*

3. Lebesgue decomposition

Definition 4. *Let m and g be two finite fuzzy measures. If $E \in \Sigma, g(E) = 0$ implies $m(E) = 0$, then we say that m is absolutely continuous with respect to g.*

Definition 5. *Let m and g be two finite fuzzy measures. If for every $\epsilon > 0$ there is a $\delta > 0$ such that $E \in \Sigma, g(E) < \delta$ implies $m(E) < \epsilon$, then we say that m is absolutely $\epsilon-$continuous with respect to g.*

Theorem 3. *Let m and g be two finite fuzzy measures such that they are continuous from above and continuous from below. If g is autocontinuous from above, then m is absolutely continuous with respect to g iff m is absolutely $\epsilon-$continuous with respect to g.*

Proof. It is obvious that if m is absolutely $\epsilon-$continuous with respect to g, then m is absolutely continuous with respect to g.
Suppose now that $E \in \Sigma, g(E) = 0$ implies $m(E) = 0$. If the theorem would not be true, then there would exist $\epsilon > 0$ and a sequence $\{E_n\}$ from Σ such that

(7) $$g(E_n) < \frac{1}{n} \quad and \quad m(E_n) > \epsilon \quad (n \in N).$$

Since g is autocontinuous from above there exists a subsequence $\{E_{n_k}\}$ of the sequence $\{E_n\}$ such that

(8) $$g(\bigcup_{i=s}^{k} E_{n_i}) < \frac{1}{s} \quad for \quad s = 1, 2, ..., k.$$

By the continuity from above of g we have

$$\lim_{s\to\infty} g\Big(\bigcup_{i=s}^{\infty} E_{n_i}\Big) = g\Big(\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} E_{n_i}\Big). \tag{9}$$

Since g is continuous from below we obtain by (8)

$$g\Big(\bigcup_{i=s}^{\infty} E_{n_i}\Big) = \lim_{k\to\infty} g\Big(\bigcup_{i=s}^{k} E_{n_i}\Big) \le \frac{1}{s}.$$

Hence by (9)

$$g\Big(\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} E_{n_i}\Big) = 0,$$

which implies

$$m\Big(\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} E_{n_i}\Big) = 0.$$

On the other hand, we obtain by the continuity from above and continuity from below of the fuzzy measure m and (7)

$$m\Big(\bigcap_{s=1}^{\infty}\bigcup_{i=s}^{\infty} E_{n_i}\Big) = \lim_{s\to\infty} m\Big(\bigcup_{i=s}^{\infty} E_{n_i}\Big) = \lim_{s\to\infty}\lim_{k\to\infty} m\Big(\bigcup_{i=s}^{k} E_{n_i}\Big) \ge m(E_{n_s}) > \epsilon.$$

Contradiction.

Theorem 4. *Let m be a null-additive fuzzy measure which is continuous from above and continuous from below. Then there exists a set A from Σ such that*

$$m(A) = \sup\{m(E), E\in\Sigma\}, \tag{10}$$

$$m(E\setminus A) = 0 \text{ and } m(E) = m(E\cap A) \quad (E\in\Sigma).$$

Proof. We shall choose a sequence $\{A_n\}$ from Σ which will generate the desired set A. Let $A_0 = \emptyset$. We take A_1 from Σ such that

$$m(A_1) = \sup\{m(E) : E\in\Sigma\}.$$

This is possible by the continuity from below of m. We choose A_2 from Σ such that

$$m(A_2) = \sup\{m(E) : E \subset X \setminus A_1\}.$$

Repeating this procedure, we choose a sequence $\{A_n\}$ such that

$$m(A_n) = \sup\{m(E) : E \subset X \setminus \bigcup_{i=0}^{n-1} A_i, E \in \Sigma\} \tag{11}$$

holds. We take $A = \bigcup_{i=0}^{\infty} A_i$. Then by the construction (10) holds. The continuity from above of m implies

$$\lim_{n\to\infty} m(E \setminus \bigcup_{i=0}^{n} A_i) = m(E \setminus A). \tag{12}$$

By (11) we obtain

$$\limsup_{n\to\infty} m(A_n) \geq \lim_{n\to\infty} m(E \setminus \bigcup_{i=0}^{n} A_i).$$

Hence by the exhaustivity of m (Proposition 1, [18]) and (12) $m(E \setminus A) = 0$. Hence by the null-additivity of m

$$m(E) = m((E \cap A) \cup (E \setminus A)) = m(E \cap A).$$

Definition 6. *Let m and g be two finite fuzzy measures defined on Σ. The fuzzy measure m is called singular with respect to $g, m \perp g$, if there exists a set A from Σ such that*

$$m(E \setminus A) = g(E) = 0 \quad (E \in \Sigma).$$

Remark. By Theorem 4, if for null-additive fuzzy measures m and g, which are continuous from above and continuous from below, $m \perp g$ holds, then we have $g \perp m$ too.

Now we have the following two theorems of Lebesgue decomposition type.

Theorem 5. *Let m and g be two null-additive fuzzy measures on Σ. Then there exist two null-additive fuzzy measures m_c and m_s such that $m_c(E) = m(E \setminus A)$ and $m_s(E) = m(E \cap A)$ for a set $A \in \Sigma$ and m_c is absolutely continuous with respect to g and m_s is singular with respect to g.*

Proof. The family

$$\Sigma_1 = \{E \in \Sigma : g(E) = 0\}$$

is a $\sigma-$ subalgebra of the $\sigma-$ ring Σ. By Theorem 4 the restriction of m on Σ_1 has a set $A \in \Sigma_1$ such that $m(E \setminus A) = 0$ and $m(E) = m(E \cap A)$ for $E \in \Sigma_1$. We take

$$m_c(E) = m(E \setminus A)$$

and

$$m_s(E) = m(E \cap A)$$

for each $E \in \Sigma$. It is easy to check that m_c and m_s are null-additive fuzzy measures and that m_c is absolutely continuous with respect to g, and m_s is singular with respect to g.

Theorem 6. *Let m and g be two autocontinuous from above fuzzy measures on Σ, which are continuous from above and from below. Then there exist two autocontinuous from above fuzzy measures m_c and m_s such that $m_c(E) = m(E \setminus A)$ and $m_s(E) = m(E \cap A)$ for a set $A \in \Sigma$ and m_c is absolutely $\epsilon-$continuous with respect to g, and m_s is singular with respect to g.*

Proof. We take the same m_c and m_s as in the proof of Theorem 5. Then by Theorem 3. m_c is absolutely $\epsilon-$continuous.

References

[1] Aumann, R.J, Shapley, L.S., Values of Non-Atomic Games, Princeton Univ. Press, 1974.

[2] Chateauneuf, A., Uncertainty aversion and risk aversion in models with nonadditive probabilities, in: Risk, Decision and Rationality, B.R. Munier (ed.), D. Reidel Publ. Comp., 1987,615-627.

[3] Choquet, G., Theory of capacities, Ann. Inst. Fourier (Grenoble), 5 (1953-1954), 131-292.

[4] Dobrakov, I., On submeasures - I, Dissertationes Math., 112, Warszawa, 1974.

[5] Drewnowski, L., Topological rings of sets, continuous set functions, integration ,I,II,III, Bull.Acad.Polon.Sci.Ser.Math.Astronom.Phys., 20 (1972),269-276, 277-286, 439-445.

[6] Drewnowski, L., On the continuity of certain non-additive set functions, Colloquium Math. 38 (1978), 243-253.

[7] Guariglia, E., On Dieudonne's Boundedness Theorem, J. Math. Anal. Appl. 145 (1990), 447-454.

[8] Dubois, D., Prade, M., A class of fuzzy measures based on triangular norms, Internat. J.Gen. System 8 (1982), 43-61.

[9] Ichihashi, H., Tanaka, M., Asai, K., Fuzzy Integrals Based on Pseudo-Additions and Multiplications, J. Math. Anal. Appl. 130 (1988), 354-364.

[10] Ling, C.M., Representation of associative functions, Publ. Math. Debrecen 12 (1965), 189 -212.

[11] Murofushi, T., Sugeno, M., Pseudo-additive measures and integrals, J.Math.Anal.Appl. 122 (1987),197-222.

[12] Pap, E., A generalization of a theorem Dieudonne for k-triangular set functions, Acta Sci. Math. 50 (1986),159-167.

[13] Pap, E., The Vitali-Hahn-Saks theorems for k-triangular set functions, Atti Sem. Mat. Fis. Univ. Modena 26 (1987), 21-32.

[14] Pap, E., On non-additive set functions, Atti. Sem. Mat. Fis. Univ. Modena 39 (1991), 345-360.

[15] Pap, E., Lebesgue and Saks decompositions of $\perp$ - decomposable measures, Fuzzy Sets and Systems 38 (1990), 345-353.

[16] Pap, E., Extension of the continuous t-conorm decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat.Fak. Ser. Mat. 20,2 (1990),121-130.

[17] Pap, E., Regular Borel t-conorm decomposable measures, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat.Fak. Ser. Mat. 20,2 (1990),113-120.

[18] Pap, E., The range of null-additive fuzzy and non-fuzzy measures, Fuzzy Sets and Systems 65 (1994), 105-115.

[19] Pap, E., Decomposable measures and applications on nonlinear partial differential equations, Rend.Circolo Mat. del Palermo Ser II 28 (1992), 387-403.

[20] Pap, E., Regular null-additive monotone set functions, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 25,2 (1995), 93-102.

[21] Schweizer, B., Sklar, A., Associative functions and abstract semigroups, Publ. Math. Debrecen 10 (1963), 69 -81.

[22] Sugeno, M., Theory of fuzzy integrals and its applications, Ph.D.Thesis,Tokyo Institute of Technology, 1974.

[23] Suzuki, H., On fuzzy measures defined by fuzzy integrals, J. Math. Anal. Appl. 132 (1988), 87-101.

[24] Suzuki, H., Atoms of fuzzy measures and fuzzy integrals, Fuzzy Sets and Systems 41 (1991),329-342.

[25] Wang, Z., The Autocontinuity of Set Function and the Fuzzy Integral, J. Math. Anal. Appl. 99 (1984).

[26] Wang, Z., On the null-additivity and the autocontinuity of a fuzzy measure, Fuzzy Sets and Systems 45 (1992), 223-226.

[27] Weber, S., $\perp$ - decomposable measures and integrals for Archimedean t-conorm, J. Math. Anal. Appl. 101 (1984), 114-138.

Received by the editors May 20, 1993.