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ON REFLEXIVITY OF A QUATERNION NORMED SPACE

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Abstract

For a two-side quaternion normed space X, we prove that the left dual space X', and similarly the second left dual space X'', are two-side quaternion Banach spaces. The corresponding property for the left quaternion normed spaces fails. Using a nonstandard construction, we succeed to embed the space X two-linearly and isometrically into the second dual space X''. Consequently, the notion of reflexivity can be introduced in a natural way in such spaces.

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1. Let $Q = \{\alpha = a + bi + cj + dk \,|\, a, b, c, d \in R\}$ be the noncommutative division ring of real quaternions. Here $i^2 = j^2 = k^2 = -1$, and $ij = -ji = k, jk = -kj = i, ki = -ik = j$. $\overline{\alpha} = a - bi - cj - dk$ will denote the conjugate of α, and $|\alpha| = \sqrt{a^2 + b^2 + c^2 + d^2}$ the absolute value of α. $R = \{\alpha \,|\, b = c = d = 0\}$ can be identified with the real field, and $C = \{\alpha \,|\, c = d = 0\}$ with the complex field. If $\alpha = a + bi + cj + dk$, then $a = \mathrm{Re}\,(\alpha)$ is called the real part of α. Every quaternion α satisfies the identity

$$\alpha = \mathrm{Re}\,(\alpha) + \mathrm{i}\,\mathrm{Re}\,(-\mathrm{i}\alpha) + \mathrm{j}\,\mathrm{Re}\,(-\mathrm{j}\alpha) + \mathrm{k}\,\mathrm{Re}\,(-\mathrm{k}\alpha).$$

If $\alpha \neq 0$, then $\alpha^{-1} = \dfrac{\overline{\alpha}}{|\alpha|^2}$. For arbitrary quaternions α and β, we have $\mathrm{Re}\,(\alpha\beta) = \mathrm{Re}\,(\beta\alpha)$.

2. We note that the quaternion Banach and Hilbert spaces have not been treated much in the literature. See for instance [3], [4], [5], and very interesting monograph [1], where other references about this subject have been cited.

In the sequel, we let $X \neq \{0\}$ be an arbitrary *two-side* quaternion normed space, which in particular has the properties

$$\begin{aligned} rx &= xr \qquad (x \in X, r \in R), \\ ||\alpha x|| &= ||x\alpha|| = |\alpha|\,||x|| \qquad (x \in X, \alpha \in Q). \end{aligned}$$

Next, let X' be the space of all bounded *left linear* functionals on X with the norm

$$||f|| = \sup\{|f(x)| : x \in X, ||x|| = 1\}\,,$$

that is *the left dual space* of X.

Define:

$$(\alpha f)(x) = f(x\alpha) \quad , \quad (f\alpha)(x) = f(x)\alpha,$$

for any $x \in X$, $f \in X'$ and $\alpha \in Q$. Then, as is easily seen, the space X' becomes a two-side quaternion Banach space. In particular, we have

$$rf = fr \qquad (f \in X', r \in R),$$

and

$$||\alpha f|| = ||f\alpha|| = |\alpha|\,||f|| \qquad (f \in X', \alpha \in Q).$$

We note that two-side quaternion spaces seem to be more convenient for our purpose, since the corresponding left dual space X' becomes also a two-side quaternion space. Otherwise, the dual space X' is only a *real* Banach space, without quaternionic structure, and no fine definition of reflexivity can be given.

Next, let X'' be the second dual space of the space X, that is the set of all bounded left linear functionals on X', with the norm

$$||F|| = \sup\{|F(g)| : g \in X', ||g|| = 1\}.$$

Then X'' is also a two-side quaternion Banach space. We note that scalar multiplication in the space X'' is introduced by

$$(\alpha F)(g) = F(g\alpha), (F\alpha)(g) = F(g)\alpha,$$

for any functional $F \in X''$, $g \in X'$ and any $\alpha \in Q$.

3. Now, we shall define, in a nonstandard way, a canonical embedding of the space X into the second dual space X''. For an arbitrary $x \in X$, define a functional F_x on X' by

$$F_x(g) = \mathrm{Re}\,(\mathrm{g(x)}) - \mathrm{i}\,\mathrm{Re}\,(\mathrm{g(xi)}) - \mathrm{j}\,\mathrm{Re}\,(\mathrm{g(xj)}) - \mathrm{k}\,\mathrm{Re}\,(\mathrm{g(xk)})$$

for any $g \in X'$. It is easily seen that the functional F_x has the following properties:

$$\begin{array}{l} F_x(g + g_1) = F_x(g) + F_x(g_1), \\ F_x(rg) = rF_x(g)\,,\ F_x(ig) = iF_x(g), \\ F_x(jg) = jF_x(g)\,,\ F_x(kg) = kF_x(g) \end{array}$$

for any $g, g_1 \in X'$ and $r \in R$. Whence we get that F_x is left linear on the dual space X'.

Besides, we have

$$\begin{array}{rcl} |F_x(g)|^2 & = & \mathrm{Re}^2(\mathrm{g(x)}) + \mathrm{Re}^2(\mathrm{g(xi)}) + \mathrm{Re}^2(\mathrm{g(xj)}) + \mathrm{Re}^2(\mathrm{g(xk)}) \\ & \le & |g(x)|^2 + |g(xi)|^2 + |g(xj)|^2 + |g(xk)|^2 \\ & \le & ||g||^2||x||^2 + ||g||^2||xi||^2 + ||g||^2||xj||^2 + ||g||^2||xk||^2 \\ & = & 4||g||^2||x||^2 \qquad (g \in X'), \end{array}$$

whence we have

$$|F_x(g)| \le 2||x||\,||g|| \qquad (g \in X'),$$

thus $||F_x|| \le 2||x||$. Therefore, $F_x \in X''$ for every $x \in X$.

Hence, the mapping $\pi\colon X \mapsto X''$ defined by $\pi(x) = F_x$ is an embedding of the space X into the second dual space X''.

Proposition 1. *The mapping π is two-linear and isometric.*

Proof. Since

$$F_{x+y}(g) = F_x(g) + F_y(g),$$

for any $g \in X'$ and $x, y \in X$, that is $F_{x+y} = F_x + F_y$, π is an additive mapping.

Next, we easily find that

$$\begin{aligned} F_{rx}(g) &= rF_x(g) = (rF_x)(g) \\ F_{ix}(g) &= F_x(gi) = (iF_x)(g), \\ F_{jx}(g) &= (jF_x)(g),\ F_{kx}(g) = (kF_x)(g) \qquad (g \in X', x \in X), \end{aligned}$$

whence

$$F_{\alpha x} = \alpha F_x \qquad (x \in X, \alpha \in Q).$$

Thus, π is a left linear mapping on X. In the proofs of the above relations we used the fact that $\mathrm{Re}\,(\alpha\beta) = \mathrm{Re}\,(\beta\alpha)$ for any two quaternions α and β.

Similarly, one can find that

$$F_{x\alpha} = F_x\alpha \qquad (x \in X, \alpha \in Q),$$

whence π is also right linear on X, thus it is two-linear on X.

We still have to prove that π is isometric, that is

$$||F_x|| = ||x|| \qquad (x \in X).$$

Since

$$\begin{aligned} |F_x(g)| &= \sqrt{\mathrm{Re}^2(g(x)) + \mathrm{Re}^2(g(xi)) + \mathrm{Re}^2(g(xj)) + \mathrm{Re}^2(g(xk))} \\ &\geq |\mathrm{Re}\,(g(x))|, \end{aligned}$$

we obviously have that

$$\begin{aligned} (1) \qquad ||F_x|| &= \sup\{|F_x(g)| : g \in X',\ ||g|| = 1\} \\ &\geq \sup\{|\mathrm{Re}\,(g(x))| : g \in X', ||g|| = 1\}. \end{aligned}$$

Now, we shall prove that

$$(2) \qquad \sup\{|\mathrm{Re}\,(\mathrm{g}(\mathrm{x}))| : \mathrm{g} \in \mathrm{X}', ||\mathrm{g}|| = 1\} = ||\mathrm{x}||.$$

We have that

$$|\mathrm{Re}\,(\mathrm{g}(\mathrm{x}))| \leq |\mathrm{g}(\mathrm{x})| \leq ||\mathrm{g}||\,||\mathrm{x}|| = ||\mathrm{x}||,$$

if $g \in X'$, $||g|| = 1$, so that

$$\sup\{|\mathrm{Re}\,(\mathrm{g}(\mathrm{x}))| : \mathrm{g} \in \mathrm{X}', ||\mathrm{g}|| = 1\} \leq ||\mathrm{x}||.$$

If $x = 0$, then (2) is obviouisly true. If $x \neq 0$, then by a consequence of the quaternion Hahn–Banach theorem, there is a functional $g \in X'$ such that $||g|| = 1$ and $g(x) = ||x||$. Hence, we obtain (2) again.

Relation (1) now gives

$$(3) \qquad ||F_x|| \geq ||x||.$$

Next, let $g_n \in X'$ $(n \in N)$ be a sequence of functionals such that $||g_n|| = 1$ for all $n \in N$ and

$$|F_x(g_n)| \to ||F_x||.$$

Since the case $F_x = 0$ is trivial, we can assume that $F_x \neq 0$. Put

$$F_x(g_n) = A_n,$$

and observe that $A_n \neq 0$ $(n \geq n_0)$. If we take $\lambda_n = |A_n|A_n^{-1}$ $(n \geq n_0)$, then $|\lambda_n| = 1$, $||\lambda_n g_n|| = 1$ $(n \geq n_0)$, and

$$F_x(\lambda_n g_n) = \lambda_n F_x(g_n) = |A_n| \to ||F_x||.$$

as $n \to \infty$. Taking $\lambda_n g_n = h_n$ $(n \geq n_0)$, we get $||h_n|| = 1$ $(n \geq n_0)$. and

$$F_x(h_n) = |A_n| = \mathrm{Re}\,(\mathrm{h_n}(\mathrm{x})) \qquad (\mathrm{n} \geq \mathrm{n_0}).$$

Since

$$\mathrm{Re}\,(\mathrm{h_n}(\mathrm{x})) = |\mathrm{A_n}| \to ||\mathrm{F_x}||$$

as $n \to \infty$, we obviously get that

$$(4) \qquad ||x|| = \sup\{|\mathrm{Re}\,(\mathrm{g}(\mathrm{x}))| : \mathrm{g} \in \mathrm{X}', ||\mathrm{g}|| = 1\} \geq ||\mathrm{F_x}||.$$

Combining (3) and (4), the last relation gives

$$||F_x|| = ||x|| \qquad (x \in X).$$

This completes the proof.

By the above proposition, the image

$$\pi(X) = \{F_x | x \in X\}$$

of the space X under the mapping π, is a left/right subspace of the Banach space X''. If $\pi(X) = X''$, X is called *reflexive* space, otherwise it is called *nonreflexive.*

For instance, every finite dimensional quaternionic normed space is reflexive, as well as every quaternion Hilbert space, and all quaternion spaces ℓ^p $(p > 1)$.

4. The main question now is the connection of the above notion of reflexivity with the corresponding notion of the *real symplectic image* X_r of the space X. We remember that real normed space X_r has the same elements and the same norm as X, while left (or right) scalar multiplication by reals is induced by real quaternions.

Denote by X'_r the real dual space of the space X_r, and by X''_r the second dual space of the space X_r. X_r is a real subspace of the real Banach space X''_r under the canonical mapping $x \mapsto T_x$ defined by $T_x(g_r) = g_r(x)\,(g_r \in X'_r)$. The space X is called *R-reflexive* if the space X_r is reflexive, thus if $X = X''_r$.

Also note that the space X'_r has the structure of a left quaternion Banach space, if we define

$$(\alpha g_r)(x) = g_r(x\alpha) \qquad (g_r \in X'_r, \alpha \in Q).$$

Proposition 2. *A two-side quaternion Banach space X is reflexive if and only if it is R-reflexive.*

Proof. (i) Assume, first, that X is a reflexive space, and consider an arbitrary functional $T \in X''_r$.

Denote by $\theta\colon X' \mapsto X'_r$ the mapping defined by

$$(\theta g)(x) = \mathrm{Re}\,(\mathrm{g(x)}) = \mathrm{g_r(x)} \qquad (\mathrm{g} \in \mathrm{X}', \mathrm{x} \in \mathrm{X}),$$

whence

$$g(x) = g_r(x) - i\, g_r(ix) - j\, g_r(jx) - k\, g_r(kx).$$

It is not difficult to see that θ is a quaternionic left linear, bijective, and isometric mapping from the space X' onto the space X'_r.

Consider the functional F on X' defined by

$$F(g) = T(g_r) - iT(ig_r) - jT(jg_r) - kT(kg_r), \tag{5}$$

where $g_r = \theta(g)$ for any $g \in X'$.

It is a routine job to see that F is left linear on the space X'. Also, since $||g|| = ||g_r||$, it is not difficult to see that F is bounded on X', and moreover $||F|| = ||T||$. Therefore, $F \in X''$.

Since X is a reflexive space, there exists an $x \in X$ such that

$$\begin{aligned} F(g) &= F_x(g) = \\ &= \mathrm{Re}\,(\mathrm{g(x)}) - \mathrm{i}\,\mathrm{Re}\,(\mathrm{g(xi)}) - \mathrm{j}\,\mathrm{Re}\,(\mathrm{g(xj)}) - \mathrm{k}\,\mathrm{Re}\,(\mathrm{g(xk)}) \\ &= g_r(x) - i\, g_r(xi) - j\, g_r(xj) - k\, g_r(xk). \end{aligned} \tag{6}$$

From relations (5) and (6), we get

$$T(g_r) = g_r(x) \qquad (g_r \in X'_r),$$

so that X_r is a reflexive space.

(ii) Conversely, assume that X_r is a reflexive space. Choose an arbitrary functional $F \in X''$, and consider the functional T on X'_r defined by

$$T(g_r) = \mathrm{Re}\,(\mathrm{F(g)}) \qquad (\mathrm{g_r} \in \mathrm{X'_r})),$$

where $g = \theta^{-1}(g_r)$.

T is obviously R-linear on X'_r, and we have $||T|| = ||F||$. Hence $T \in X''_r$. Since X_r is a reflexive space, there is an $x \in X_r = X$ such that

$$T(g_r) = g_r(x) \qquad (g_r \in X'_r).$$

Next, since

$$F(g) = \mathrm{Re}\,(\mathrm{F(g)}) - \mathrm{i}\,\mathrm{Re}\,(\mathrm{F(ig)}) - \mathrm{j}\,\mathrm{Re}\,(\mathrm{F(jg)}) - \mathrm{k}\,\mathrm{Re}\,(\mathrm{F(kg)}),$$

and

$$\mathrm{Re}\,(\mathrm{F}(\alpha \mathrm{g})) = \mathrm{T}(\theta(\alpha \mathrm{g})) = \mathrm{T}(\alpha\theta(\mathrm{g})) = (\alpha \mathrm{g_r})(\mathrm{x}) = \mathrm{g_r}(\mathrm{x}\alpha),$$

for any $\alpha \in Q$, we find that

$$\begin{aligned} F(g) &= g_r(x) - ig_r(xi) - jg_r(xj) - kg_r(xk) \\ &= \mathrm{Re}\,(\mathrm{g}(\mathrm{x})) - \mathrm{i}\,\mathrm{Re}\,(\mathrm{g}(\mathrm{xi})) - \mathrm{j}\,\mathrm{Re}\,(\mathrm{g}(\mathrm{xj})) - \mathrm{k}\,\mathrm{Re}\,(\mathrm{g}(\mathrm{xk})) \\ &= F_x(g) \qquad (g \in X'). \end{aligned}$$

Hence, X is a reflexive space.

This completes the proof.

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