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ON GENERIC MULTI-ALGEBRAS

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Abstract

We prove some axiomatizability results about the so-called generic multi-algebras, which are connected with operator of generating subalgebras in some universal algebra.

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1. Introduction

Multi-algebras are known in the literature under several different names: multi-algebras, poly-algebras, hyper-algebras. There are more than one hundred papers concerning some special type of multi-algebras, such as multi-groups, multi-rings, multi-semigroups, multi-groupoids... In papers [3], [4], [6], various aspects of the general theory of multi-algebras are studied. There is, of course, a special interest to study multi-algebras which are in some way connected with the basic constructions in universal algebra. In the present paper we are investigating multi-algebras which are connected with the operator of generating subalgebras in some universal algebra.

2. Preliminaries

Let us recall some basic notions. Let A be a non-empty set. *A multi-operation of arity n* is a mapping from A^n into the family $\mathcal{P}(A)$ of all subsets of A. Roughly speaking, a *multi-algebra* is a non-empty set with some multi-operations. Precisely, we have the following definition.

Definition 1. *Let $\mathcal{F}$ be a* type *i.e. a non-empty disjoint union of some sets $\mathcal{F}_n$, $n \in \mathbf{N}$. The elements of $\mathcal{F}$ we call* functional symbols, *and the* arity *of $f \in \mathcal{F}_n$ is n (we write $ar(f) = n$). A* multi-algebra of type $\mathcal{F}$ *is an ordered pair $\mathcal{A} = (A, \mathcal{F}^{\mathcal{A}})$, where A is a non-empty set (the* basis *of $\mathcal{A}$), and $\mathcal{F}^{\mathcal{A}} = \{f^{\mathcal{A}} : f \in \mathcal{F}\}$ is a family of multi-operations on A, such that the, so-called,* interpretation *$f^{\mathcal{A}}$ of the functional symbol $f \in \mathcal{F}_n$ is an n-ary multi-operation on A.*

The notion of *$\mathcal{F}$-terms over X*, ($X \neq \emptyset$, $\mathcal{F} \cap X = \emptyset$), is defined in the usual way. The *interpretations* of $\mathcal{F}$-terms on a multi-algebra $\mathcal{A} = (A, \mathcal{F}^{\mathcal{A}})$ are multi-functions such that:

(i) If t is a variable, then the interpretation is the unary multi-function $t^{\mathcal{A}} : A \to \mathcal{P}(A)$, such that $t^{\mathcal{A}}(a) = \{a\}$, for all $a \in A$;

(ii) If terms $t_1(\bar{x}_1), \ldots, t_n(\bar{x}_n)$ are interpreted by multi-functions $t_n^{\mathcal{A}}(\bar{x}_n)$, $\ldots, t_n^{\mathcal{A}}(\bar{x}_n)$, and $f \in \mathcal{F}_n$, then term $t = f(t_1(\bar{x}_1), \ldots, t_n(\bar{x}_n))$ has the interpretation $t^{\mathcal{A}}$, such that

$$t^{\mathcal{A}}(\bar{a}_1, \ldots, \bar{a}_n) = \bigcup \{f^{\mathcal{A}}(c_1, \ldots, c_n) : \; c_i \in t^{\mathcal{A}}(\bar{a}_i), \; 1 \leq i \leq n\}.$$

Depending on the language of the theory of multi-algebras of type $\mathcal{F}$, we consider the following two possibilities:

1) Calculus I: the pure predicate calculus (of type $\mathcal{F}$), in which the atomic formulas are of the form $t_1 \approx t_2$ (t_1 and t_2 are $\mathcal{F}$-terms), where $\approx$ interprets as the usual equality of two sets.

2) Calculus II: the pure predicate calculus (of type $\mathcal{F}$), in which instead of equality of two terms of type $\mathcal{F}$, the atomic formulas have the form $t_1 \subseteq t_2$. In this case, we interpret $\subseteq$ in these atomic formulas as the usual set-theoretical inclusion.

Of course, calculus II is "richer" than calculus I.

If $\mathcal{A} = (A, \mathcal{F}^{\mathcal{A}})$ is a universal algebra, $X \subseteq A$, then with $\langle X \rangle_{\mathcal{A}}$ we denote the basic set of the subalgebra of $\mathcal{A}$, generated by X.

Definition 2. *Let $\mathcal{A} = (A, \mathcal{F}^{\mathcal{A}})$ be a universal algebra. The* generic multi-algebra corresponding to $\mathcal{A}$ *is the multi-algebra $G(\mathcal{A})$ of type $\Sigma = \{p_1, p_2, \ldots, p_n \ldots\}$, $ar(p_n) = n$, $n \geq 1$, with basis A, such that for all $n \geq 1$, $a_1, \ldots, a_n \in A$,*

$$p_n^{G(\mathcal{A})}(a_1, \ldots, a_n) = \langle \{a_1, \ldots, a_n\} \rangle_{\mathcal{A}}.$$

In the sequel, Σ will always denote the type $\{p_1, \ldots, p_n, \ldots\}$, where $ar(p_n) = n$, for $n \geq 1$.

3. Results

The first natural question which arises is whether the class of all generic multi-algebras can be described (in the class of all multi-algebras of type Σ) by some formulas (identities, quasi-identities,...) of calculus I or II.

Theorem 1. *The class of all generic multi-algebras is not axiomatizable in calculus I.*

Proof. We have to prove that there is no set of first-order formulas in calculus I which would describe the class of all generic algebras within the class of all multi-algebras of type Σ.

Let $\mathcal{A}$ be the algebra $(\mathbf{Z}; g, h)$, where $\mathbf{Z}$ is the set of all integers, and for any $a \in \mathbf{Z}$,

$$g(a) = a + 1, \quad h(a) = a - 1.$$

Then, in generic multi-algebra $G(\mathcal{A})$ we have that

$$p^{G(\mathcal{A})}(a_1, \ldots, a_n) = \mathbf{Z}.$$

Let $\mathbf{N}$ denote the set of all natural numbers, and define $\mathcal{A}'$ as the multi-algebra of type Σ, with basis $\mathbf{Z}$, such that for all $n \geq 1$, and $a_1, \ldots, a_n \in \mathbf{Z}$

$$p_n^{\mathcal{A}'}(a_1, \ldots, a_n) = \mathbf{N}.$$

Of course, $\mathcal{A}'$ is not a generic multi-algebra, because, for example, $\{-1\} \not\subseteq p_1^{\mathcal{A}'}(-1)$. On the other hand, multi-algebras $G(\mathcal{A})$ and $\mathcal{A}'$ satisfy the same atomic formulas in calculus I. Precisely, if $t_1 = t_1(x_1, \ldots, x_n)$ and $t_2 = t_2(x_1, \ldots, x_n)$ are some terms of type Σ, and $a_1, \ldots, a_n \in \mathbf{Z}$, we have

$$t_1^{G(\mathcal{A})}(a_1, \ldots, a_n) = t_2^{G(\mathcal{A})}(a_1, \ldots, a_n) \Leftrightarrow t_1^{\mathcal{A}'}(a_1, \ldots, a_n) = t_2^{\mathcal{A}'}(a_1, \ldots, a_n).$$

In this way, multi-algebras $G(\mathcal{A})$ and $\mathcal{A}'$ satisfy the same first-order formulas in calculus I. Hence, the class of all generic multi-algebras is not axiomatizable in calculus I. □

Now we shall prove that the class of all generic multi-algebras is axiomatizable in calculus II. Moreover, we shall give explicitly a set of defining axioms.

Definition 3. *Let (P1), (P2), (P3) be the following lists of formulas of type Σ in calculus II:*

(P1) $x_i \subseteq p_n(x_1, \ldots, x_i, \ldots, x_n)$, *for all* $n \geq 1$ *and* $1 \leq i \leq n$;

(P2) $p_n(x_1, \ldots, x_n) \subseteq p_k(y_1, \ldots, y_k)$, *for all* $n, k \geq 1$ *and all* $x_1, \ldots, x_n$, $y_1, \ldots, y_k$ *such that* $\{x_1, \ldots, x_n\} \subseteq \{y_1, \ldots, y_k\}$;

(P3) $p_n(x_1, \ldots, x_{n-1}, p_m(y_1, \ldots, y_m)) = p_{n+m-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m)$ *for all* $n, m \geq 1$.

Theorem 2. *A multi-algebra $\mathcal{B}$ of type Σ is a generic multi-algebra iff $\mathcal{B}$ identically satisfies the system of formulas (P1), (P2), (P3). So, the class of all generic multi-algebras is axiomatizable in calculus II.*

Proof. Of course, every generic multi-algebra satisfies (P1), (P2), (P3). Conversely, let $\mathcal{B}$ be a multi-algebra with basis A, such that (P1), (P2), (P3) hold. Then, there is an algebra $\mathcal{A}$ such that $\mathcal{B} = G(\mathcal{A})$. Namely, let us define a mapping $\varphi : \mathcal{P}(A) \to \mathcal{P}(A)$ in the following way: for any $X \subseteq A$,

$$\varphi(X) = \bigcup\{p_n^{\mathcal{B}}(a_1, \ldots, a_n) : \ a_1, \ldots, a_n \in X, \ n \geq 1\}.$$

Then, φ will be an algebraic closure operator on A, i.e. for any $X, Y \subseteq A$ we have:

(C1) $X \subseteq \varphi(X)$,

(C2) $\varphi(\varphi(X)) = \varphi(X)$,

(C3) $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$,

(C4) $\varphi(X) = \bigcup\{\varphi(Z) |\ Z \subseteq X$ and Z is finite $\}$.

Namely, (C1) follows from (P1), (C3) follows from (P2), and (C1) follows from the definition of φ. To prove (C2), let us note that the inclusion $\varphi(X) \subseteq \varphi(\varphi(X))$ holds because of (C1) and (C3). Conversely, $\varphi(\varphi(X)) \subseteq \varphi(X)$, since (using (P3)) we have:

$$\begin{aligned}\varphi(\varphi(X)) &= \bigcup\{p_n^{\mathcal{B}}(a_1,\ldots,a_n) : a_i \in \varphi(X),\ 1 \le i \le n,\ n \ge 1\} = \\ &= \bigcup\{p_n^{\mathcal{B}}(a_1,\ldots,a_n) : a_i \in \bigcup\{p_m(b_1,\ldots,b_m) : b_j \in X, \\ &\quad 1 \le j \le m,\ m \ge 1\},\ 1 \le i \le n,\ n \ge 1\} \subseteq \\ &\subseteq \{p_k(c_1,\ldots,c_k) : c_i \in X,\ 1 \le i \le k,\ k \ge 1\} = \varphi(X).\end{aligned}$$

According to Theorem of Birkhoff and Frink ([1]), there is an algebra $\mathcal{A}$, with basis A, such that for all $X \subseteq A$, $\varphi(X) = \langle X \rangle_{\mathcal{A}}$. As $\varphi(\{a_1,\ldots,a_n\}) = p_n^{\mathcal{B}}(a_1,\ldots,a_n)$, we have that $\mathcal{B} = G(\mathcal{A})$. □

As a consequence of the last theorem, we can prove the compactness theorem for the class of all generic multi-algebras in calculus II.

Corollary 1. *Let T be a set of first-order formulas of type Σ in calculus II. If every finite subset of T holds on some generic multi-algebra, then there is a generic multi-algebra which satisfies T.*

Proof. Let $\mathcal{C}$ be a multi-algebra of type Σ, with the basis C. We define an algebraic system $\Pi(\mathcal{C})$ with the basis $C \cup \mathcal{P}(C)$, in the language $\Sigma' = \Sigma \cup \{P, \in\}$, where P and $\in$ are relational symbols of arity 1 and 2, respectively, in the following way:

- for $n \ge 1$, $p_n \in \Sigma_n$, $a_1,\ldots,a_n \in C$, $b \in \mathcal{P}(C)$, $p_n^{\Pi(\mathcal{C})}(a_1,\ldots,a_n) = b$ iff $p_n^{\mathcal{C}}(a_1,\ldots,a_n) = b$;
- for $n \ge 1$, $p_n \in \Sigma_n$, and if some elements from $a_1,\ldots,a_n$ are from $\mathcal{P}(C)$, then
$$p_n^{\Pi(\mathcal{C})}(a_1,\ldots,a_n) = \emptyset;$$
- $\Pi(\mathcal{C}) \models P(a)$ iff $a \in \mathcal{P}(C)$;

- $\in$ interprets in $\Pi(\mathcal{C})$ as the usual set-theory relation $\in$.

Let Π be the class of all algebraic systems which are isomorphic to $\Pi(\mathcal{C})$, for some multi-algebra $\mathcal{C}$ of type Σ. Note that for every $\mathcal{D} \in \Pi$ we can find a unique multi-algebra $\mathcal{C}$, such that $\mathcal{D} \cong \Pi(\mathcal{C})$. Obviously, the class Π of algebraic systems is elementary (i.e. axiomatizable in the first-order predicate calculus). Since the class of all generic multi-algebras is axiomatizable in calculus II (T2), we have that class

$$\Pi_G = \{\mathcal{A} : \mathcal{A} \cong \Pi(G(\mathcal{B})),\ \mathcal{B} \text{ is a universal algebra}\}$$

is elementary too. This fact, and the Compactness Theorem for the usual first-order predicate calculus imply the claim of our corollary. □

References

[1] Birkhoff, G., Frink, O., Representations of lattices by sets, Trans. Amer. Math. Soc., v. 64 (1948), 299-316.

[2] Bruck, R. H., A Survey of Binary Systems, Springer-Verlag, Berlin-Gottingen-Heidelberg, 1958.

[3] Čupona, G., Madarász, Sz. R., Free poly-algebras, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23, 2 (1993), 245-261.

[4] Čupona, G., Madarász, Sz. R., On poly-algebras, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 21, 2 (1991), 141-156.

[5] Dresher, M., Ore, M., Theory of multigroups, Amer. J. Math. 60 (1938), 705-733.

[6] Vaš, L., Madarász, Sz. R., A note about multi-algebras, power-algebras and identities, Proceed. of IX Conf. on Appl. Math., Novi Sad, 1995, 147-153.

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