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ON CARISTI'S FIXED POINT THEOREM IN F-TYPE TOPOLOGICAL SPACES

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Abstract

A generalization of Caristi's fixed point theorem from [2] and [3] is obtained.

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1. Introduction

In [2], the notion of an $F-$type topological space is introduced and some fixed point theorems and variational principles in such kind of spaces are proved. In this paper, using Hick's theorem [5] a generalization of Theorem 3.1 from [2] is obtained and a fixed point theorem of Caristi's type in fuzzy metric spaces is proved.

2. Preliminaries

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}^+$ the set of non-negative real numbers, $\mathbf{N}$ the set of natural numbers, and $D = (D, \prec)$ a directed set. In [2], the following definition is introduced.

Definition. *A topological space (X, τ) is said to be $F-$type if it is Hausdorff and for each $x \in X$ there exists a neighbourhood base $\mathcal{U}_x = \{U_x(\lambda, t);\ \lambda \in D,\ t > 0\}$ of x with the following properties:*

(F-1) If $y \in U_x(\lambda, t)$ then $x \in U_y(\lambda, t)$.

(F-2) $U_x(\lambda, t) \subset U_x(\mu, s)$, for $\lambda \prec \mu, t \leq s$.

(F-3) For every $\lambda \in D$ there exists $\mu \in D$ with $\lambda \prec \mu$ such that

$$U_x(\mu, t_1) \cap U_y(\mu, t_2) \neq \emptyset \Rightarrow y \in U_x(\lambda, t_1 + t_2).$$

(F-4) $X = \bigcup_{t>0} U_x(\lambda, t)$, for each $\lambda \in D$ and $x \in X$.

In [2], it is proved that the topology τ of an $F-$type topological space (X, τ) can be generated by a family $M = \{d_\lambda;\ \lambda \in D\}$ of quasi-metrics on X so that (P-1) - (P-4) hold, where:

(P-1) $d_\lambda(x, y) = 0$, for all $\lambda \in D \iff x = y$.

(P-2) $d_\lambda(x, y) = d_\lambda(y, x)$, for all $\lambda \in D,\ x, y \in X$.

(P-3) $d_\lambda(x, y) \leq d_\mu(x, y)$, for all $\lambda \prec \mu$ and $x, y \in X$.

(P-4) For every $\lambda \in D$ there exists $\mu \in D$ with $\lambda \prec \mu$ such that

$$d_\lambda(x, y) \leq d_\mu(x, z) + d_\mu(z, y),$$

for all $x, y, z \in X$.

This means that $\tau = \tau_M$ where, for every $x \in X$, $\mathcal{B}_x = \{B_x(\lambda, t);\ \lambda \in D,\ t > 0\}$ is a neighbourhood base for x in τ_M and for $\lambda \in D$ and $t > 0$

$$B_x(\lambda, t) = \{y;\ y \in X,\ d_\lambda(x, y) < t\}.$$

As was proved in [2], the Hausdorff topological vector spaces and some classes of Menger probabilistic metric spaces belong to the class of $F-$type topological spaces. The class of $F-$ type topological spaces contains the class of fuzzy metric spaces (X, d, L, R) [6], where $\lim_{a \to 0_+} R(a,a) = 0$ and $\lim_{t \to \infty} d(x,y)(t) = 0$, for every $x, y \in X$.

A fuzzy number $u : \mathbf{R} \to [0,1]$ is said to be **normal** if there exists $t_0 \in \mathbf{R}$ such that $u(t_0) = 1$ and **convex** if for every $t_1, t_2 \in \mathbf{R}$ and every $\mu \in [0,1]$

$$u(\mu t_1 + (1-\mu)t_2) \geq \min\{u(t_1), u(t_2)\}.$$

All upper semicontinuous, normal, convex fuzzy numbers are denoted by $\mathcal{E}$. Let

$$\mathcal{E}^+ = \{u;\ u \in \mathcal{E}, u(t) = 0, \text{ for all } t < 0\}.$$

$\alpha-$level cuts $[u]_\alpha = \{t;\ t \in \mathbf{R},\ u(t) \geq \alpha\}$ $(\alpha \in (0,1])$ of a fuzzy number u are nonempty closed intervals $[a^\alpha, b^\alpha]$, where the values $-\infty$ and ∞ are admissible.

Let $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments which satisfy $L(0,0) = 0,\ R(1,1) = 1$.

Let X be a nonempty set, $d : X \times X \to \mathcal{E}^+$ and for every $\alpha \in (0,1]$ and $(x,y) \in X \times X$

$$[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)],\ \alpha \in (0,1].$$

The quadruple (X, d, L, R) is called **fuzzy metric space** and d a **fuzzy metric** if (i) - (iii) hold, where:

(i) For every $x, y \in X$

$$d(x,y) = I_{\{0\}} \iff x = y$$

where

$$I_{\{0\}}(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

(ii) For every $x, y \in X,\ d(x,y) = d(y,x)$.

(iii) For every $x, y, z \in X$

a) $d(x,z)(s+t) \geq L(d(x,y)(s), d(y,z)(t))$, whenever $s \leq \lambda_1(x,y),\ t \leq \lambda_1(y,z)$ and $s + t \leq \lambda_1(x,z)$,

b) $d(x,z)(s+t) \leq R(d(x,y)(s), d(y,z)(t))$, whenever $s \geq \lambda_1(x,y)$, $t \geq \lambda_1(y,z)$ and $s+t \geq \lambda_1(x,z)$.

The following theorem can be proved by the results from [4], [6].

Theorem A. *Let (X,d,L,R) be a fuzzy metric space such that $\lim_{a\to 0_+} R(a,a) = 0$, $\lim_{t\to\infty} d(x,y)(t) = 0$. Then (X,τ) is an $F-$type topological space where $\tau = \tau_M$, $M = \{\rho'_\alpha;\ \alpha \in [0,1)\}$, $\rho'_\alpha = \rho_{1-\alpha}$ $(\alpha \in [0,1))$.*

Proof. From the condition $\lim_{t\to\infty} d(x,y)(t) = 0$ it follows that $\rho_\alpha(x,y) < \infty$, for every $\alpha \in (0,1]$ and $x,y \in X$. Since ρ_α is nonincreasing in α, ρ'_α is nondecreasing in $\alpha \in [0,1)$, then (P-3) is satisfied. (P-1) and (P-2) are obviously satisfied. In [4], it is proved that for $\alpha,\mu \in (0,1]$:

$$R(\mu,\mu) < \alpha \Rightarrow \rho_\alpha(x,y) \leq \rho_\mu(x,z) + \rho_\mu(z,y),$$

for every $x,y,z \in X$. Since $\lim_{a\to 0_+} R(a,a) = 0$, for every $\alpha \in (0,1]$, there exists $\mu < \alpha$ such that $R(\mu,\mu) < \alpha$. Hence

$$\rho_\alpha(x,y) \leq \rho_\mu(x,z) + \rho_\mu(z,y)$$

and since $\mu < \alpha$, if $\alpha = 1-\alpha'$, $\mu = 1-\mu'$, we have that $\alpha' < \mu'$ and

$$\rho'_{\alpha'}(x,y) \leq \rho'_{\mu'}(x,z) + \rho'_{\mu'}(z,y)$$

which means that (P-4) holds.

3. A Caristi's type fixed point theorem

The following theorem has been proved by Hicks [5].

Theorem B. *Let $[U]$ be a quasi-uniform structure for a set X, $\preceq$ a partial order for X and $\phi : X \to [b,\infty]$ $(b \in \mathbf{R})$ is not identically ∞. Suppose that the following conditions hold:*

a) $S(x,\preceq) = \{y;\ y \in X,\ x \preceq y\}$ is complete for every $x \in X$ i.e. every nondecreasing Cauchy sequence from $S(x,\preceq)$ is converging in $S(x,\preceq)$.

b) ϕ is nonincreasing.

c) For each $U \in [U]$, there exists $r > 0$ such that for every $(x_1, x_2) \in X \times X$

$$(x_1 \preceq x_2) \wedge (\phi(x_1) - \phi(x_2) < r) \Rightarrow (x_1, x_2) \in U.$$

Then for each $x \in X$ with $\phi(x) < \infty$, there exists $x_0 \in X$ with $\phi(x_0) < \infty$ such that $x \preceq x_0$ and x_0 is maximal on $(X, \preceq)$.

Using Theorem B we shall prove the following theorem.

Theorem 1. *Let (X, τ) be a complete $F-$type topological space, $\{d_\lambda;\ \lambda \in D\}$ a generating family of quasi-metrics for τ, $\phi : X \to [b, \infty]$ ($b \in \mathbf{R}$) a proper lower semicontinuous mapping and $k : D \to (0, \infty)$ a nonincreasing function. Then, for each $x \in X$ with $\phi(x) < \infty$ there exists an element $x_0 \in X$ such that:*

(i) $\phi(x_0) < \infty$.

(ii) For every $\lambda \in D$

$$d_\lambda(x, x_0) \leq k(\lambda)(\phi(x) - \phi(x_0)).$$

(iii) For every $y \in X \setminus \{x_0\}$, $\phi(y) = \infty$ or $\phi(y) < \infty$ and there exists $\lambda \in D$ such that

$$k(\lambda)(\phi(x_0) - \phi(y)) < d_\lambda(x_0, y).$$

Proof. It is easy to prove that $\preceq$ is a partial order on X, where

$$(1) \qquad x \preceq y \iff (x = y) \vee ((\phi(x) < \infty) \wedge (\phi(y) < \infty)$$
$$\wedge (d_\lambda(x, y) \leq k(\lambda)(\phi(x) - \phi(y))), \text{ for every } \lambda \in D).$$

We shall prove that all the conditions of Theorem B are satisfied.

It is obvious that $\preceq$ is a partial order. We shall show that ϕ is non-increasing. Suppose that $x \preceq y$. If $x = y$, then $\phi(x) = \phi(y)$ and if $\phi(x) < \infty$, $\phi(y) < \infty$ and for every $\lambda \in D$

$$0 \leq d_\lambda(x, y) \leq k(\lambda)(\phi(x) - \phi(y))$$

it follows that $\phi(y) \leq \phi(x)$. The condition c) from Theorem B holds. Indeed, if $U \in [U]$ is of the form

$$U = U(\lambda, t) = \{(x, y); (x, y) \in X \times X,\ d_\lambda(x, y) < t\},$$

then for $r = \dfrac{t}{k(\lambda)}$ we have the implication

$$(x_1 \preceq x_2) \wedge (\phi(x_1) - \phi(x_2) < r) \Rightarrow d_\lambda(x_1, x_2) < t.$$

It remains to be proved that $S(x, \preceq)$ is closed, for every $x \in X$. We may assume that $\phi(x) < \infty$. Let $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ be a generalized sequence from $S(x, \preceq)$ such that $\lim\limits_{\alpha \in \mathcal{A}} x_\alpha = z$. We have to prove that $x \preceq z$, which means that $z \in S(x, \preceq)$. Since $x_\alpha \in S(x, \preceq)$ and $\phi(x) < \infty$ we have that $\phi(x_\alpha) < \infty$, for every $\alpha \in \mathcal{A}$ and that for every $\alpha \in \mathcal{A}$ and $\lambda \in D$

$$d_\lambda(x_\alpha, x) \leq k(\lambda)(\phi(x) - \phi(x_\alpha)).$$

We shall prove that for every $\lambda \in D$

$$d_\lambda(x, z) \leq k(\lambda)(\phi(x) - \phi(z)).$$

Let $\lambda \in D$ and choose $\mu \in D$ with $\lambda \prec \mu$ such that for all $u, v, w \in X$

$$d_\lambda(u, v) \leq d_\mu(u, w) + d_\mu(w, v).$$

Since $\lim\limits_{\alpha \in \mathcal{A}} x_\alpha = z$ there exists, for every $\epsilon > 0$, $\alpha(\mu, \epsilon) \in \mathcal{A}$ such that

$$d_\mu(x_\alpha, z) < \epsilon, \quad \text{for every} \quad \alpha \geq \alpha(\mu, \epsilon).$$

Then

$$\begin{aligned} d_\alpha(x, z) &\leq d_\mu(x, x_\alpha) + d_\mu(x_\alpha, z) \\ &< k(\mu)(\phi(x) - \phi(x_\alpha)) + \epsilon, \quad \text{for every} \quad \alpha \geq \alpha(\mu, \epsilon). \end{aligned}$$

Using the lower semicontinuity of ϕ we obtain that

$$d_\lambda(x, z) \leq k(\mu)(\phi(x) - \liminf_{\alpha \in \mathcal{A}} \phi(x_\alpha)) + \epsilon$$

$$\leq k(\mu)(\phi(x) - \phi(z)) + \epsilon \leq k(\lambda)(\phi(x) - \phi(z)) + \epsilon.$$

Since ϵ is an arbitrary positive number the proof is complete.

Hence, all the conditions of Theorem B are satisfied and there exists $x_0 \in X$ so that (i), (ii) and (iii) hold.

The following theorem generalizes Theorem 3.1 from [2].

Theorem 2. *Let (X,τ) be a complete $F-$type topological space, $\{d_\lambda;\ \lambda \in D\}$ a generating family of quasi-metrics for τ, $\phi : X \to [b,\infty]$ $(b \in \mathbf{R})$ be a proper lower semicontinuous mapping, $f : X \to X$ and $k : D \to (0,\infty)$ a nonincreasing function so that $\phi(x) < \infty$ implies $\phi(fx) < \infty$ and for every $x \in X$ such that $\phi(x) < \infty$*

$$d_\lambda(x, fx) \leq k(\lambda)(\phi(x) - \phi(fx)) \text{ for every } \lambda \in D. \tag{2}$$

Then, for every $x \in X$ such that $\phi(x) < \infty$ there exists a fixed point $\bar{x} \in X$ of f such that $\phi(\bar{x}) < \infty$ and for every $\lambda \in D$

$$d_\lambda(x, \bar{x}) \leq k(\lambda)(\phi(x) - \phi(\bar{x})). \tag{3}$$

If $y \neq \bar{x}$ and $\phi(y) < \infty$, then there exists $\lambda \in D$ such that

$$k(\lambda)(\phi(\bar{x}) - \phi(y)) < d_\lambda(\bar{x}, y).$$

Proof. Let $x \in X$ be such that $\phi(x) < \infty$. From Theorem 1 it follows that there exists $\bar{x} \in X$ such that $\phi(\bar{x}) < \infty$ and that (3) holds for every $\lambda \in D$. From Theorem 1 and the relation $\phi(f\bar{x}) < \infty$ it follows that $\bar{x} \neq f\bar{x}$ implies the existence of $\lambda \in D$ such that

$$k(\lambda)(\phi(\bar{x}) - \phi(f\bar{x})) < d_\lambda(\bar{x}, f\bar{x}).$$

This contradicts to (2). Hence $\bar{x} = f\bar{x}$.

Corollary. *Let (X, d, L, R) be a complete fuzzy metric space such that $\lim_{a\to 0_+} R(a,a) = 0$ and for every $x, y \in X$, $\lim_{t\to\infty} d(x,y)(t) = 0$.*

Let $\phi : X \to [b,\infty] (b \in \mathbf{R})$ be a proper lower semicontinuous mapping, $f : X \to X$ and $h : (0,1] \to (0,\infty)$ a nondecreasing mapping so that $\phi(x) < \infty$ implies $\phi(fx) < \infty$ and for every $x \in X$ such that $\phi(x) < \infty$

$$\rho_\alpha(x, fx) \leq h(\alpha)(\phi(x) - \phi(fx)) \text{ for every } \alpha \in (0,1].$$

Then, for every $x \in X$ such that $\phi(x) < \infty$ there exists a fixed point $\bar{x} \in X$ of f such that $\phi(\bar{x}) < \infty$ and for every $\alpha \in (0,1]$

$$\rho_\alpha(x, \bar{x}) \leq h(\alpha)(\phi(x) - \phi(\bar{x})).$$

If $y \neq \bar{x}$ and $\phi(y) < \infty$, then there exists $\alpha \in (0,1]$ such that

$$h(\alpha)(\phi(\bar{x}) - \phi(y)) < \rho_\alpha(\bar{x}, y).$$

Proof. If $\rho'_\alpha = \rho_{1-\alpha}$ and $k(\alpha) = h(1-\alpha), \alpha \in [0,1)$, then we can apply Theorem 2 for $D = [0,1)$, $d_\alpha = \rho'_\alpha(\alpha \in D)$.

If $h(\alpha) = 1$, $\alpha \in (0,1]$ from Corollary we obtain Caristi's fixed point theorem from [4].

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