

MINIMAX THEOREMS IN CONVEX SPACES

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Abstract

We apply our basic coincidence theorem in [6] to obtain new forms of minimax theorems and a saddle point result. Our results extend the works of Arandjelović [1] and Sion [7], and represent variants of the works of Ha [3], Komornik [5], and Komiya [4].

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1. Introduction

Recently ([6]) we introduced a "better" admissible class \mathfrak{B} of multimaps and a basic coincidence theorem for \mathfrak{B} as well as a matching theorem and a KKM theorem. Those results are subsequently applied to the problems related to a generalized minimax inequality in [7] and to extensions of monotone sets in [8].

In the present paper, we apply the basic coincidence theorem in [6] to obtain new forms of minimax theorems and a saddle point result.

From our basic theorem (Theorem 1), we deduce first a particular coincidence theorem (Theorem 2) extending Fan's result [2]. Then, from Theorem 2, we deduce a minimax theorem (Theorem 3), which is a variant of a result of Komornik [5] and extends a saddle point result due to Arandjelović [1] and Sion [7]. Finally, we obtain another minimax theorem (Theorem 4) involving acyclic sets, which is a variant of the results of Ha [3], Komornik [5], and Komiya [4].

2. Coincidence Theorems

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of another set Y . Note that $y \in T(x)$ is equivalent to $x \in T^-(y)$, and $T(A) = \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

A *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls will be called *polytopes*.

For topological spaces X and Y , a multimap $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each open subset G of Y , the set $\{x \in X : Tx \subset G\}$ is open in X ; and *compact* whenever $T(X)$ is relatively compact in Y . Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

In our previous work [6], we introduced a “better” admissible class \mathfrak{B} of maps defined on a convex space X as follows:

$$T \in \mathfrak{B}(\mathfrak{X}, \mathfrak{Y}) \iff \text{for any polytope } P \text{ in } X \text{ and any } f \in \mathbf{C}(\mathbf{F}(\mathbb{P}), \mathbb{P}), \\ \text{the composite } f(T|_P) : P \multimap P \text{ has a fixed point.}$$

We give some examples of \mathfrak{B} as follows:

$$t \in \mathbf{C}(\mathfrak{X}, \mathfrak{Y}) \iff t \text{ is a continuous (single-valued) function.}$$

$$T \in \mathbf{K}(\mathfrak{X}, \mathfrak{Y}) \iff T \text{ is a u.s.c. map with nonempty compact convex values, where } Y \text{ is a convex space.}$$

$$T \in \mathbf{V}(\mathfrak{X}, \mathfrak{Y}) \iff T \text{ is an acyclic map; that is, a u.s.c. map with compact acyclic values.}$$

$$T \in \Phi(X, Y) \iff T(x) \text{ is nonempty convex for each } x \in X \text{ and } T^-(y) \\ \text{is open for each } y \in Y, \text{ where } Y \text{ is a convex space.}$$

There are many other examples of \mathfrak{B} ; see [6].

The following is due to the author [6, Theorem 1]:

Theorem 1. *Let X be a convex space, Y a Hausdorff space, and $T, S : X \rightarrow Y$ maps satisfying*

- (1) $T \in \mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ is compact;
- (2) for each $y \in T(X)$, $S^{-}(y)$ is convex; and
- (3) $\{\text{Int } S(x) : x \in X\}$ covers the closure $\overline{T(X)}$.

Then T and S have a coincidence point $x_0 \in X$; that is, $T(x_0) \cap S(x_0) \neq \emptyset$.

From Theorem 1, we obtain the following theorem, which shows that there is another subclass of \mathfrak{B} bigger than Φ :

Theorem 2. *Let X be a convex space, Y a Hausdorff convex space, and $F, G : X \rightarrow Y$ maps such that*

- (1.1) F is compact, $F(x)$ is convex for each $x \in X$, and $X = \bigcup\{\text{Int } F^{-}(y) : y \in Y\}$; and
- (1.2) $G^{-}(y)$ is convex for each $y \in F(X)$ and $\overline{F(X)} = \bigcup\{\text{Int } G(x) : x \in X\}$.

Then F and G have a coincidence point.

Proof. In view of Theorem 1, it suffices to show that $F \in \mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$. Let P be a polytope in X . Since P is compact, there exists a finite subset $\{y_1, y_2, \dots, y_n\} \subset Y$ such that $P \subset \bigcup_{i=1}^n \text{Int } F^{-}(y_i)$. Let $\{\lambda_i\}_{i=1}^n$ be the partition of unity subordinated to the cover of P . Define $h : P \rightarrow Y$ by

$$h(x) = \sum_{i=1}^n \lambda_i(x)y_i = \sum_{i \in N_x} \lambda_i(x)y_i \quad \text{for } x \in P,$$

where

$$i \in N_x \iff \lambda_i(x) \neq 0 \implies x \in \text{Int } F^{-}(y_i) \subset F^{-}(y_i).$$

Then $y_i \in Fx$ for each $i \in N_x$. Clearly, h is continuous and, by (1.1), $h(x) \in \text{co}\{y_i : i \in N_x\} \subset Fx$ for each $x \in P$. Therefore, h is a continuous

selection of $F|_P$. Since $h : P \rightarrow h(P) \subset F(P)$, for any $f : \mathbb{C}(\mathbb{F}(\mathbb{P}), \mathbb{P})$, the composite $fh : P \rightarrow P$ is a continuous selection of $f(F|_P) : P \rightarrow P$ and has a fixed point by the Brouwer fixed point theorem. Hence, $F \in \mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$. Now the conclusion follows from Theorem 1. \square

Remark 1. *Theorem 2 improves the result of Fan [2].*

3. New Minimax Theorems

In this section, Z denotes a *complete linearly ordered space*; that is, a linearly ordered set whose every subset has a least upper bound. Examples are the extended real line $\overline{\mathbf{R}}$, the extended Euclidean space $\overline{\mathbf{R}}^n$, and any compact (in the Euclidean topology) subset of \mathbf{R}^n with respect to the lexicographic order; see Komornik [5].

For a topological space X , a function $f : X \rightarrow Z$ is said to be *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] whenever $\{x \in X : f(x) > z\}$ [resp. $\{x \in X : f(x) < z\}$] is open in X for each $z \in Z$.

If X is compact and $f : X \rightarrow Z$ is l.s.c., then there exists an $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$. For any family $\{f_i\}_{i \in I}$ of l.s.c. functions $f_i : X \rightarrow Z$, the function $\sup_{i \in I} f_i$ is also l.s.c. (see [5]).

The following is the main result of this section:

Theorem 3. *Let X be a convex space, Y a Hausdorff compact convex space, and $f : X \times Y \rightarrow Z$ a function. Suppose that*

- (1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap (a, b) \neq \emptyset$;*
- (2) *$f(x, \cdot)$ is l.s.c. on Y and $\{y \in Y : f(x, y) < s\}$ is convex for each $x \in X$ and $s \in U$; and*
- (3) *$f(\cdot, y)$ is u.s.c. on X and $\{x \in X : f(x, y) > s\}$ is convex for each $y \in Y$ and $s \in U$.*

Then

$$\sup_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Proof. Since $f(x, \cdot)$ is l.s.c., $p(x) = \min_{y \in Y} f(x, y)$ exists for each $x \in X$. Since $q(y) = \sup_{x \in X} f(x, y)$ is l.s.c. for each $y \in Y$, $q(y_0) = \min_{y \in Y} q(y)$ exists. Note that

$$p(x) = \min_{y \in Y} f(x, y) \leq f(x, y) \leq \sup_{x \in X} f(x, y) = q(y)$$

for all $x \in X$ and $y \in Y$. Therefore, we have

$$\sup_{x \in X} p(x) \leq \min_{y \in Y} q(y).$$

Suppose that the equality does not hold. Then, there exists an $s \in U$ such that

$$\sup_{x \in X} p(x) < s < \min_{y \in Y} q(y).$$

We define the multimaps $T, S : X \multimap Y$ by

$$T(x) = \{y \in Y : f(x, y) < s\} \text{ and } S(x) = \{y \in Y : f(x, y) > s\}$$

for $x \in X$. Then $T(x)$ is nonempty and convex by (2), and $S(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover,

$$T^-(y) = \{x \in X : f(x, y) < s\} \text{ and } S^-(y) = \{x \in X : f(x, y) > s\}$$

for $y \in Y$. Then $T^-(y)$ is open since $f(\cdot, y)$ is u.s.c., and $S^-(y)$ is nonempty and convex by (3). Now, by applying Theorem 2, there exists an $x_0 \in X$ such that $T(x_0) \cap S(x_0) \neq \emptyset$. This contradicts

$$f(x_0, a) < s < f(x_0, b) \text{ for each } a \in T(x_0) \text{ and } b \in S(x_0).$$

This completes our proofs. \square

Remark 2. If $U = Z$, then Theorem 3 is a consequence of Komornik [5, Theorem 2] for interval spaces with different proof.

Corollary 1. Under the hypothesis of Theorem 3, further if X is compact, then f has a saddle point.

Proof. Since $f(x, \cdot)$ and $f(\cdot, y)$ are l.s.c. and u.s.c., resp., $p(x) = \min_{y \in Y} f(x, y)$ and $q(y) = \max_{x \in X} f(x, y)$ exist for each $x \in X$ and $y \in Y$. Since p is u.s.c. on X and q is l.s.c. on Y , $\max_{x \in X} p(x) = p(x_0)$ and $\min_{y \in Y} q(y) = q(y_0)$ for some $x_0 \in X$ and $y_0 \in Y$. Then (x_0, y_0) is a saddle point by Theorem 3. This completes our proof. \square

Remark 3. The corollary reduces to the results of Arandjelović [1, Theorem 3] whenever $Z = \mathbf{R}$, which extends the Sion minimax theorem [9].

The following new minimax theorem is a variant of Theorem 3:

Theorem 4. Let X be a convex space, Y a Hausdorff compact space, and $f : X \times Y \rightarrow Z$ a l.s.c. function such that

- (1) there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap [a, b) \neq \emptyset$;
- (2) for each $s \in U$ and $y \in Y$, $\{x \in X : f(x, y) > s\}$ is convex; and
- (3) for each $s \in U$ and $x \in X$, $\{y \in Y : f(x, y) \leq s\}$ is acyclic.

Then

$$\sup_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Proof. As in the proof of Theorem 3, we have

$$\sup_{x \in X} p(x) \leq \min_{y \in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $s \in U$ such that

$$\sup_{x \in X} p(x) \leq s < \min_{y \in Y} q(y).$$

We define the multimaps $T, S : X \multimap Y$ by

$$T(x) = \{y \in Y : f(x, y) \leq s\} \text{ and } S(x) = \{y \in Y : f(x, y) > s\}$$

for $x \in X$. Then $T(x)$ is nonempty by the definition of $p(x)$ and closed since $f(x, \cdot)$ is l.s.c. for each $x \in X$. On the other hand, $S(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover, for each $y \in Y$,

$$S^-(y) = \{x \in X : f(x, y) > s\}$$

is nonempty and convex by (2).

Consider the graph of T

$$\text{Gr}(T) = \{(x, y) \in X \times Y : f(x, y) \leq s\}.$$

Since f is l.s.c., $\text{Gr}(T)$ is closed in $X \times Y$. Since Y is compact, T is u.s.c. Note that each $T(x)$ is acyclic by (3). Hence T is an acyclic map.

Therefore by Theorem 1 for \mathbb{V} instead of \mathfrak{B} , there exists an $x_0 \in X$ such that $T(x_0) \cap S(x_0) \neq \emptyset$. This leads a contradiction as in the proof of Theorem 3. \square

Remark 4. 1. In case we replace the acyclicity in (3) by convexity and if $U = Z = \mathbf{R}$, then Theorem 4 is a particular form of Ha [3, Theorem 4].

2. If we replace the acyclicity in (3) by convexity and if $U = Z$, then Theorem 4 follows from Komornik [5, Theorem 3] with different proof.

3. Komiya [4, Theorem 3] obtained a saddle point theorem whenever $U = Z = \mathbf{R}$, f is continuous, and the acyclicity in (3) is replaced by convexity in Theorem 4 under an extra restriction.

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