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THE CUBIC SPLINE DIFFERENCE SCHEME ON NON-UNIFORM MESH

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Abstract

The linear singularly perturbed boundary value problem of the second order is considered. The spline difference schemes applied on such problem give the system of the linear equations with the tridiagonal matrix of L-form. For small values of the parameter the matrix loses L-form and the system becomes unstable. At the same time the truncation error goes to infinity when small parameter goes to zero. For obtaing uniform stability and simple structure of the matrix a fitting factor of the polynomial form is introduced. Schishkin mesh is used in order to obtain uniform convergence.

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1. Difference Scheme

Let us consider the following singularly perturbed problem

$$(1)\qquad \begin{cases} Ly = -\varepsilon^2 y'' + p(x)y = f(x), & x \in I = (0,1), \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, \end{cases}$$

where $\varepsilon \in (0, \varepsilon_0), \varepsilon_0 \ll 1$, is a small perturbation parameter. The functions p and f are given and we assume

$$p, f \in C^2(I), \quad p(x) \geq \beta^2 > 0.$$

It is known that the problem (1) has a unique solution y, which in general displays boundary layers at x=0 and x=1. Under the above assumptions the exact solution has the form ([1]):

$$y(x) = v(x) + g(x), \tag{2}$$

where

$$|g^{(j)}(x)| \leq M, \ j = 1, 2, 3, 4$$

$$|v^{(j)}(x)| = M\varepsilon^{-j}(e^{-x\frac{b}{\varepsilon}} + e^{(x-1)\frac{b}{\varepsilon}}), \ j = 1, 2, 3, 4.$$

$p(x) \geq \beta^2 > 0$, $b = \min(\beta, 1)$. In [4] a difference scheme via cubic spline on a non-uniform mesh Δ,

$$\Delta : 0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$$

is derived. The scheme has the form:

$$\begin{cases} Ru_j = Qf_j, \ j = 1(1)n-1, \\ u_0 = \alpha_0, \quad u_n = \alpha_1, \end{cases} \tag{3}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

and

$$\begin{cases} r_j^- = -(1 - \frac{h_{j-1}^2 p_{j-1}}{6\varepsilon^2})\frac{1}{h_{j-1}}, \ r_j^+ = -(1 - \frac{h_j^2 p_{j+1}}{6\varepsilon^2})\frac{1}{h_j}, \\ r_j^c = (1 + \frac{h_{j-1}^2 p_j}{3\varepsilon^2})\frac{1}{h_j} + (1 + \frac{h_j^2 p_j}{3\varepsilon^2})\frac{1}{h_j}, \\ q_j^- = \frac{h_{j-1}}{6\varepsilon^2}, \ q_j^+ = \frac{h_j}{6\varepsilon 2}, \ q_j^c = \frac{h_j + h_{j-1}}{3\varepsilon^2}, \end{cases} \tag{4}$$

$h_j = x_j - x_{j+1}$ and $q_j = q(x_j)$. The corresponding matrix has the $L - form$

when $h_{j-1}^2 p_{j-1} \leq 6\varepsilon^2$ and $h_j^2 p_{j+1} \leq 6\varepsilon^2$. This is a consequence of the maximum principle which the problem (1) satisfies. For small ε these inequalities require forbiden small step of the mesh. For the clasical difference scheme the situation is diferent. The scheme has the form (3)([6]) where

$$r_j^- = \frac{1}{h_{j-1}\bar{h}_{j-1}},\ r_j^+ = \frac{1}{h_j\bar{h}_{j-1}},\ r_j^c = -\frac{2}{h_j h_{j-1}}, \tag{5}$$

$$q_j^- = 0,\ q_j^+ = 0,\ q_j^c = 1.$$

and the matrix has $L - form$ independently of ε. But the truncation error for bouth spline and central difference scheme becomes infinity when ε goes to zero. In [6] the central difference scheme (3) (5) is considered and the Shishkin mesh is used in order to avoid the mentioned dificulties. The error estimate of the form $O(n^{-2}ln^2 n)$ is proved. In this paper we applay the Shishkin mesh on a spline approximation for the same purpose. The stability of the scheme is achieved by introducing the fitting factor near the second derivative (artifical viskosity). In [8] a spline difference scheme with fitting factor is derived. The fitting factor has the exponential form which povide the uniform convergence. The scheme has the form (3) with

$$\begin{cases} r_j^- = -(1 - \frac{h_{j-1}^2 p_{j-1}}{6\sigma_{j-1}})\frac{1}{h_{j-1}},\ r_j^+ = -(1 - \frac{h_j^2 p_{j+1}}{6\sigma j+1})\frac{1}{h_j}, \\ r_j^c = (1 + \frac{h_{j-1}^2 p_j}{3\sigma_j})\frac{1}{h_j} + (1 + \frac{h_j^2 p_j}{3\sigma_j})\frac{1}{h_j}, \\ q_j^- = \frac{h_{j-1}}{6\sigma_{j-1}},\ q_j^+ = \frac{h_j}{6\sigma_{j+1}},\ q_j^c = \frac{h_j + h_{j-1}}{3\sigma_j} \end{cases} \tag{6}$$

We wont to obtain the second order of the convergence which is the property of spline collocation and L-form of the matrix. Because of that by analyse of truncation error we can see that σ_j should be determined so that $r^- \geq 0$, $r^+ \geq 0$ and $\sigma_j - \varepsilon = O(h_j^2)$ for small ε. Thus, $\sigma_j = h_j^2 p_j/(6\rho_j)$, where $\rho_j = 1 + O(h_j^2)$ and $\rho_j \geq 1$ satisfies the requirements. The constants in the asymptotic equalities must be independent of ε and h_j. In [8] ρ_j is dtermined so that the truncation error for boundary layer functions vanishes. Here, we use the simpler fitting factor since the mesh has the special structure which made it possible for us to avoid difficulties connected with boundary layer functions. Our choise is:

$$
(7) \qquad \begin{cases} \sigma_j = h_j^2 p_j/6, \ 0 \leq x_j \leq 1/2, \\ \sigma_j = h_{j-1}^2 p_j/6, \ 1/2 \leq x_j \leq 1. \end{cases}
$$

Throughout the paper M denotes any positive constant that may take different values in different formulas, but that are allways independent of ε and discretization mesh.

2. Construction of the mesh

In the literature, several types of special graded meshses have been introduced for singularly perturbed two-point boundary value problems (see [3], [2], [5]). The mesh from [5] is piecewise equidistant and consequently much simpler than the other meshes. We shall use that mesh. Given a positive integer n divisible by 4. We divide the interval $[0,1]$ into the three subintervalas

$$
[0,\delta], \qquad [\delta, 1-\delta], \qquad [1-\delta, 1].
$$

We use equidistant meshes on each of these subintervals, with $1+n/4$ points in each of $[0,\delta]$ and $[1-\delta,1]$, and $1+n/2$ points in $[\delta,1-\delta]$. Let $b = min\{\beta, 1\}$ and $\delta = min\{1/4, 4b^{-1}\varepsilon \ln n\}$. Let

$$
i_0 = n/4, x_{i_0} = \delta, \ x_{n-i_0} = 1-\delta,
$$

and

$$
h_i = x_{i+1} - x_i = 4\delta n^{-1}; \ i = 1, 2, .., i_0, n - i_0 + 1, ..., n
$$

and $h_i = 2(1-2\delta)n^{-1}$ otherwise.

We assume that $\delta = 4b^{-1}\varepsilon \ln n$ since in the oposite case the method can be analysed using standard techniques. Thus we have that

$$
h_i = 16b^{-1}\varepsilon n^{-1} \ln n, \ i = 1, 2, .., i_0, n - i_0 + 1, ..., n
$$

and

$$
h_i = 2(1-2\delta)n^{-1}, \ n^{-1} \leq h_i \leq 2n^{-1}, \ i = i_0 + 1, .., n - i_0.
$$

3. Convergence results

We solve problem (1) by using scheme (3),(6),(7) on the mesh specifed in the previous section. Then $r^-_{n-i_0+1} < 0$ and $r^+_{i_0-1} < 0$. The others r^-_j and r^+_j are equal to zero. In that way the system (3),(6),(7) reduces to the following simple form:

$$
(8) \qquad \begin{cases} u_0 = \alpha_0, \\ u_i = \frac{1}{r^c_i} Qf_i, \; i = 1, 2, ..., i_0 - 2, \\ r^c_{i_0-1} u_{i_0-1} + r^+_{i_0-1} u_{i_0} = Qf_{i_0-1}, \\ u_i = \frac{1}{r^c_i} Qf_i, \; i = i_0, ..., n - i_0, \\ r^-_{n-i_0+1} u_{n-i_0} + r^c_{n-i_0+1} u_{n-i_0+1} = Qf_{n-i_0+1}, \\ u_i = \frac{1}{r^c_i} Qf_i, \; i = n - i_0 + 2, ..., n - 1, \\ u_n = \alpha_1. \end{cases}
$$

The following theorem holds.

Theorem 1. *Let* $p, f \in C^2(I)$, $p(x) \geq \beta^2 > 0$. *Let* u_j *be the solution of the system (8). Then*

$$|y(x_i) - u_i| \leq M n^{-2} ln^2 n.$$

Proof. The solution $y(x) \in C^4(I)$. The estimate of differences $e_j = y_j - u_j$ we will obtain via the expresion

$$(9) \qquad Re_j = \tau_j(y) = \tau_j(g) + \tau_j(v).$$

The truncation error $\tau_i(y) = Ry_i - Q(Ly)_i$ has the form:

$$\tau_i(y) = \phi_{2,i+1}/h_i - \phi_{2,i}/h_{i-1} + \phi_{1,i}$$

where

$$\phi_{2,i} = \psi_{0,i} + h^2_{i-1}\left(\frac{\eta_{i-1}}{3\sigma_{i-1}} + \frac{\eta_i}{6\sigma_i} - \psi_{2,i}/6\right),$$

$$\phi_{1,i} = \psi_{1,i} + \frac{h_{i-1}}{2}(\frac{\eta_{i-1}}{\sigma_{i-1}} + \frac{\eta_i}{\sigma_i} - \psi_{2,i},$$

$$\psi_{k,i} = y^{(4)}(\theta_{k,i})h_{i-1}^{(4-k)}/(4-k)!,\ x_i \le \theta_{k,i} \le x_{i+1},$$

where $\eta_i = y_i''(\sigma_i - \varepsilon^2)$. According to the exact solution and the estimate $|\eta_i| \le Mh_i^2$ we have

$$|\tau_i(g)| \le M(\varepsilon n^{-1}\ln n + n^{-1}), i = 1,2,...,n-1.$$

Further, using $|\eta_i| \le Mh_i^2\varepsilon^{-2}$ we have

$$|\tau_i(v)| \le Mn^{-1}\ln n/\varepsilon, i = 1,2,...i_0 - 1,$$

$$|\tau_i(v)| \le Mn^{-1}\ln n/\varepsilon, i = n - i_0 + 1,...,n-1,$$

$$|\tau_i(v)| \le Mn^{-1}, i = i_0 + 1,...,n - i_0 - 1.$$

Since $|r_i^c| = |\frac{3f_{i-1}h_i + h_i^2 + 2h_{i-1}^2}{h_i^2 h_{i-1}}| \le M/h_{i-1}$ from the second group of equations (8) we obtain that theorem holds for $i = 1,...,i_0 - 2$.

For index $i = i_0$ we have

$$\tau_i(v) = r_i^c v_i - q_i^- F_i - q_i^c F_i - q_i^+ F_i,$$

where $F_i = -\varepsilon^2 v_i'' + p_i v_i$. The corresponding estimate is

$$|\tau_i(v)| \le M\frac{e^{-x_{i_0}b/\varepsilon}}{h_{i-1}} \le M\frac{n^{-2}}{h_{i-1}}.$$

From the expresion of r_i^c and fourth group of equations in (8) we obtain state of the theorem. After that, from the third group of equations (8) we get estimate for $i = i_0 - 1$. Finaly, we proved the state for the first half of the interval I. The proof for the second half of the interval is the same.

4. Numerical results

In this section we present some numerical experiments using the schemes described in previous theorem. Our example is taken from [1].

Example 1.

k	n							
	16	32	64	128	256	512	1024	
3	1.72(−2)	6.24(−3)	2.14(−3)	7.11(−4)	2.32(−4)	7.29(−5)	2.34(−5)	E_n
			1.46	1.54	1.58	1.62	1.67	*Ord*
4	1.72(−2)	6.24(−3)	2.14(−3)	7.10(−4)	2.32(−4)	7.32(−5)	2.4(−5)	E_n
			1.46	1.54	1.59	1.61	1.66	*Ord*
5	1.72(−2)	6.23(−3)	2.14(−3)	7.10(−4)	2.32(−4)	7.54(−4)	2.50(−5)	E_n
			1.46	1.54	1.59	1.62	1.62	*Ord*
6	1.72(−2)	6.24(−3)	2.14(−3)	7.10(−4)	2.32(−4)	7.53(−5)	2.63(−5)	E_n
			1.46	1.55	1.59	1.60	1.61	*Ord*
7	1.72(−2)	6.24(−3)	2.14(−3)	7.10(−4)	2.32(−5)	7.54(−5)	2.84(−5)	E_n
			1.46	1.54	1.59	1.61	1.62	*Ord*
8	1.72(−2)	6.24(−3)	2.14(−3)	7.17(−4)	2.31(−5)	7.53(−5)	3.19(−5)	E_n
			1.46	1.55	1.59	1.59	1.61	*Ord*
9	1.72(−2)	6.24(−3)	2.14(−3)	7.20(−4)	2.31(−5)	7.54(−5)	3.36(−5)	E_n
			1.46	1.55	1.57	1.64	1.62	*Ord*
10	1.72(−2)	6.24(−3)	2.15(−3)	7.10(−4)	2.43(−4)	8.37(−5)	4.27(−5)	E_n
			1.46	1.54	1.60	1.54	1.54	*Ord*
11	1.72(−2)	6.24(−3)	2.15(−3)	7.10(−4)	2.48(−4)	7.54(−5)	5.56(−5)	E_n
			1.46	1.54	1.60	1.52	1.71	*Ord*
12	1.72(−2)	6.24(−3)	2.13(−3)	7.24(−4)	2.52(−4)	1.03(−4)	6.87(−5)	E_n
			1.46	1.54	1.56	1.52	1.28	*Ord*
13	1.72(−2)	6.24(−3)	2.14(−3)	7.45(−4)	2.61(−4)	1.00(−4)	9.59(−5)	E_n
			1.46	1.54	1.52	1.51	1.38	*Ord*
14	1.72(−2)	6.30(−3)	2.18(−3)	7.24(−4)	2.96(−4)	1.27(−4)	1.31(−4)	E_n
			1.45	1.53	1.59	1.29	1.22	*Ord*
15	1.72(−2)	6.33(−3)	2.20(−3)	7.45(−4)	3.22(−4)	1.63(−4)	1.67(−4)	E_n
			1.44	1.52	1.56	1.21	.98	*Ord*
16	1.72(−2)	6.23(−3)	2.14(−3)	8.39(−4)	2.48(−4)	2.86(−4)	2.78(−4)	E_n
			1.46	1.54	1.35	1.75	error	*Ord*
17	1.72(−2)	6.33(−3)	2.14(−3)	7.33(−4)	4.57(−4)	3.64(−4)	4.21(−4)	E_n

Table 1. Scheme (8).

$$-\varepsilon^2 y'' + y + \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x = 0, \qquad x \in [0,1],$$

$$u(0) = 0, \qquad u(1) = 0.$$

The exact solution has the form

$$y(x) = \frac{e^{\frac{-x}{\varepsilon}} + e^{\frac{x-1}{\varepsilon}}}{1 + e^{\frac{-1}{\varepsilon}}} - \cos^2 \pi x.$$

We denote by E_n the maximum of $|y(x_j) - u_j|, j = 0(1)n$. Here $[u_0, u_1, \dots, u_{n+1}]^{\mathsf{T}}$ is corresponding numerical solution to the system (8)(7). Also, we define in the usual way the order of convergence Ord for two succesive values of n with respective errors E_n and E_{2n}:

$$Ord = \frac{\log E_n - \log E_{n_2}}{\log n_2 - \log n},$$

where $n_2 = 2n$. Different values of $\varepsilon^2 = 2^{-11} * 2^{-k}$ and n are considered.

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