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A HIGHER ORDER APPROXIMATION TO A SINGULAR PERTURBATION PROBLEM

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Abstract

We analyse and numerically study spline difference methods applied to a singularly perturbed convection-diffusion two-point boundary value problem whose solution has a single boundary layer. The method is derived by collocation with piecewise exponential splines from $C^1[0,1]$ on an regular mesh. The errors at the grid points is bounded by $Ch^4/(\varepsilon^2 + h^2)$, C is a constant independent of small parameter ε (multiplying the highest derivative) and mesh size h.

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1. Introduction

We shall consider difference schemes for the solution of the following model differential equation:

$$(1.1)\qquad Ly \equiv \varepsilon y''(x) + p(x)y'(x) - d(x)y(x) = f(x), \qquad x \in (0,1)$$

$$By \equiv (y(0), y(1)) = (\alpha_0, \alpha_1)$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, and p, d, f are assumed to be sufficiently smooth.

It is well known that the solution of this kind of problem has boundary layer behaviour for small ε, and thus the derivatives of the solution are not uniformly buonded for $\varepsilon \in (0, \varepsilon_0]$. Without loss of generality we shall assume for $x \in [0,1]$:

$$(1.2)\qquad p(x) \geq p > 0, d(x) \geq d, \quad b^2 + 4\varepsilon d > 0.$$

Under these hypothesis L satisfies a maximum principle and (1.1) has a unique solution $y(x)$, (see [4]).

When ε is small, such model problems describe physical situation in which convective forces dominated diffusive forces. This circumstancie arises, for example, in the modelling of Navier-Stokes flow with large Reynolds numbers or heat transport problems with large Peclet numbers. So it is important to devise methods for their numerical solution. Many such methods have been proposed in the literature (see [1] and [6] for a survey).

The major problem in the numerical solution of (1.1) is to find a numerical approximation method which is uniformly accurate in ε. So we are interested in finite difference techniques for solving (1.1), (1.2) that satisfy criterion known as *uniform convergence* or *uniform accuracy*, that is

$$(1.3)\qquad \|u^h - y\| \leq Ch^r.$$

Here $r > 0$ is the convergence order, u^h is the approximate solution, y is exact solution, C is generic constant independent of ε and h. The error estimates we report are given in the discrete L^∞ norm.

The standard methods on a uniform mesh do not belong to this class. The introduction o fitted meshes or fitted schemes overcomes theese difficulties.

We concentrate on methods which do not use graded meshes in the boundary layer regions. So far strenuous efforts have made to develop figh-order numerical methods (see for example [2] and [5]). Both methods use piecewise polynomial approximations of the coefficients of (1.1). In [8] a high order scheme is derived by linear combination of scheme from the class which is obtained by piecewise constant approximations of the coefficients of the equation (1.1). Proposed method has a simpler form.

In this paper we will present a new scheme from the class obtained by collocating equation (2.2) with piecewise exponential splines from $C^1[0,1]$ and a new high order method obtained by suitable linear combination of the schemes. The errors at the grid points of the proposed method are bounded by $Ch^4/(\varepsilon^2+h^2)$.The analysis of the behaviour of the schemes for small ε gives the posibility for modifications of the method wich give better results.

The numerical results exhibiting the performances of the scheme are presented and discussed in the paper.

2. Derivation of the schemes

Let for a given integer n a uniform partition of the interval [0,1] be given by a regular mesh Δ^n

$$\Delta^n: \quad 0 = x_0 < x_1 < \dots < x_n = 1 \tag{2.1}$$

with

$$x_j = jh, \quad j = 0, \dots, n, \quad h = 1/n.$$

Let $u^h = (u_0, u_1, \dots, u_n)$ denote a discrete approximation on this mesh to the solution y of (1.1), and let p_j, $p_{j\pm1/2}$ denote $p(x_j)$, $p(x_j \pm h/2)$ for any function p defined on the mesh. Further, let a vector $\rho = (\rho_0, \dots, \rho_{n-1})$ be given with components $\rho_j > 0$, $j = 0, \dots, n-1$.

The set of all piecewise exponential splines e(x) defined by

$$e(x) \in span\{1, x, exp(\rho_j x), exp(-\rho_j x)\}, \quad x \in [x_j, x_{j+1}]$$

and $e(x) \in C^k[0,1]$ is denoted by $ESp(k,\rho)$. Since for $\rho_j \to 0$, $j = 0, \dots, n-1$ the cubic spline arises, $ESp(k,0)$ is now a set of cubic splines.

For $[x_j, x_{j+1}]$, $j = 0, ..., n-1$, set (see [3])

(2.2) $e(x) = e_j(x) = u_j + hm_jt + a_j(ch\mu_jt - 1)/\rho_j + l_j(sh\mu_jt - \mu_j)/\rho_j,$

with $t = (x - x_j)/h$, $\mu_j = h\rho_j$, ρ_j, $j = 0, ..., n-1$ are tension parameters.

Let us define the auxiliary equation to equation (1.1):

$$\tilde{L}y \equiv \varepsilon y''(x) + \tilde{p}y'(x) - \tilde{d}y(x) = \tilde{f}, \tag{2.3}$$

$$\tilde{B}y \equiv (y(0), y(1)) = (\alpha_0, \alpha_1),$$

$\tilde{p}, \tilde{d}, \tilde{f}$ are piecewise polinomial approximations to the functions $p(x), d(x)$ and $f(x)$.

Let $\tilde{p} = \tilde{p}_j$, $\tilde{d} = \tilde{d}_j$, $\tilde{f} = \tilde{f}_j$, for $x \in [x_j, x_{j+1}]$, $j = 0, ..., n-1$, where $\tilde{p}_j, \tilde{d}_j, \tilde{f}_j$ are constants. Introducing ρ_j in the proper way:

$$\rho_j = \tilde{p}_j/\varepsilon, \ j = 0, ..., n-1,$$

we specify $e(x)$ by collocating at n points , i.e. by enforcing

(2.4) $\tilde{L}e_j(x) = \tilde{f}_j,$ for $x = x_j$ and $x = x_{j+1}$, $j = 0, ..., n-1,$

and requiring $e(x)$ to satisfy the boundary conditions (1.2). Conditions (2.3) give that parameters $a_j = -l_j$ for $j = 0, ..., n-1$, the spline e(x) then has a simpler form:

(2.5) $e(x) = e_j(x) = u_j + hm_jt + a_j(exp(-\rho_j(x-x_j)) + \rho_j(x-x_j) - 1).$

The continuity condition $e(x) \in ESp(1, \rho)$ give the following family of difference schemes :

$$\varepsilon h^{-2}Ru_j = Q(f)_j, \quad j = 1, ..., n-1, \tag{2.6}$$

$$u_0 = \alpha_0, \quad u_n = \alpha_1,$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1},$$

$$Qf_j = q_j^- f_j^- + q_j^+ f_j^+,$$

where $p^- = \tilde{p}_{j-1}$, $p^+ = \tilde{p}_j$ for every fixed $j = 1, ..., n-1$. d^-, d^+, f^-, f^+ are defined similarly.

When it is clear from the context, the j subscripts will be omitted.

For $d(x) \equiv 0$ we get

$$(2.7) \qquad \begin{cases} q^- = a(\mu^-), \quad q^+ = b(\mu^+) \\ r^- = c(\mu^-), \quad r^+ = s(\mu^+), \quad r^c = -r_1^- - r_1^+, \\ \mu^\pm = \rho^\pm h, \quad \rho^\pm = p^\pm/\varepsilon, \end{cases}$$

where

$$s(t) = t/(1 - exp(-t)), \quad c(t) = texp(-t)/(1 - exp(-t)),$$

$$a(t) = (1 - c(t))/t, \quad b(t) = (s(t) - 1)/t.$$

By determining

$$p^\pm = (p(x_{j\pm1}) + p(xj))/2, \; d^\pm = (d(x_{j\pm1}) + d(xj))/2, \; f^\pm = (f(x_{j\pm1}) + f(x_j))/2$$

we obtain the well-known El-Mistikawy and Werle (EMW) scheme. When we take

$$p^\pm = p(x_{j\pm1/2}) \; , \; d^\pm = d(x_{j\pm1/2}) \; , \; f^\pm = f(x_{j\pm1/2})$$

we obtain the improved (IEMW) scheme that was analysed in [8]. Both schemes, EMW and IEMW, are second order uniformly accurate in ε (see [1],[9],[10]).

In [11] is presented and analysed in detail a new scheme (called IIEMW scheme) which is obtained by determing

$$(2.8) \qquad \begin{aligned} p^\pm &= (p(x_{j\pm1}) + 4p(x_{j\pm1/2}) + p(xj))/6, \\ d^\pm &= (d(x_{j\pm1}) + 4d(x_{j\pm1/2}) + d(xj))/6, \\ f^\pm &= (f(x_{j\pm1}) + 4f(x_{j\pm1/2}) + f(x_j))/6. \end{aligned}$$

The numerical results for small ε show the superiority of the IIEMW scheme in comparison to EMW and IEMW schemes.

In this paper we shall analyse a scheme that is obtained as a linear combination :

$$(2.9) \qquad 2IEMW \quad - \quad IIEMW.$$

The scheme (2.9) has the form:

$$\varepsilon h^{-2} R_1 u_j = Q_1 f_j \tag{2.10}$$

where

$$R_1 u_j = r_1^- u_{j-1} + r_1^c u_j + r_1^+ u_{j+1}$$

$$Q_1 f_j = q_1^- f_{j-1} + q_1^{-1/2} f_{j-1/2} + q_1^c f_j + q_1^{+1/2} f_{j+1/2} + q_1^+ f_{j+1}, \quad j = 1(1)n-1,$$

$$u_0 = \alpha_0, \quad u_1 = \alpha_1,$$

with

$$\left\{\begin{array}{l} q_1^- = -a(\mu_1^-)/6, \quad q_1^{-1/2} = 2a(\mu_2^-) - 4a(\mu_1^-)/6, \\ q_1^c = -a(\mu_1^-)/6 - b(\mu_1^+)/6, \\ q_1^{+1/2} = 2b(\mu_2^+) - 4b(\mu_2^+)/6, \quad q_1^+ = -b(\mu_1^+)/6 \\ r_1^- = 2c(\mu_2^-) - c(\mu_1^-), \ r_1^+ = 2s(\mu_2^+) - s(\mu_1^+), \ r_1^c = -r_1^- - r_1^+, \\ \mu_1^\pm = \frac{(p_{j\pm1} + p_j + 4p_{j\pm1/2})h}{6\varepsilon}, \quad \mu_2^\pm = \frac{p(x_{j\pm1/2})h}{\varepsilon}. \end{array}\right. \tag{2.11}$$

3. Accuracy of the Scheme

Let us analyse the truncation error $\tau_j^1(y)$ of the scheme (2.9) which can be presented as the linear combination of the truncation errors $\tau_j^I(y)$ and $\tau_j^{II}(y)$ of schemes *IEMW* and *IIEMW*.

The truncation error $\tau_j(y)$ of the family(2.6):

$$\tau_j(y) = Ry_j - Q(Ly_j),$$

can be written in the form

$$\tau_j(y) = T_{j0} y_j + T_{j1} y_j' + T_{j2} y_j'' + T_{j3} y_j''' + R_{j4}(y),$$

where $T_{j0} = T_{j1} = 0$ for both *IEMW* and *IIEMW* schemes.

The corresponding to *IEMW* scheme truncation error is:

$$\tau_j^I(y) = T_{j2}^I y_j'' + T_{j3}^I y_j''' + R_{j4}^I(y),$$

where

$$T_{j2}^I = \frac{\varepsilon}{2}(r^- + r^+) - \varepsilon(q^- + q^+) + \frac{h}{2}(p^- q^- - p^+ q^+),$$

$$T_{j3}^{I}=\frac{\varepsilon h}{6}(r^{+}-r^{-})+\frac{h}{2}\varepsilon(q^{-}-q^{+})-\frac{h^{2}}{8}(p^{+}q^{+}+p^{-}q^{-}),$$

$$R_{j4}^{I}(y)=T_{jr}^{I}+T_{jq}^{I},$$

$$T_{jr}^{I}=\frac{r^{-}\varepsilon}{h^{2}}R_{3}(x_{j},x_{j-1},y)+\frac{s(r^{+})\varepsilon}{h^{2}}R_{3}(x_{j},x_{j+1},y),$$

$$T_{jq}^{I}=-q^{-}\varepsilon R_{1}(x_{j},x_{j-1/2},y'')-q^{+}\varepsilon R_{1}(x_{j},x_{j+1/2},y'')-$$
$$-q^{-}p^{-}R_{2}(x_{j},x_{j-1/2},y')-q^{+}p^{+}R_{2}(x_{j},x_{j+1/2},y'),$$

where

$$R_{n}(a,b,g)=\frac{1}{n!}\int_{a}^{b}(b-s)^{n}g^{(n+1)}(s)ds=g^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!},\quad a\leq\xi\leq b.$$

The truncation error for the $IIEMW$ scheme is:

$$\tau_{j}^{II}(y)=T_{j2}^{II}y_{j}''+T_{j3}^{II}y_{j}'''+R_{j4}^{II}(y),$$

where

$$T_{j2}^{II}=(\varepsilon/2)(r^{-}+r^{+})-\varepsilon(q^{-}+q^{+})$$
$$+(h/6)(q^{-}(p_{j-1}+2p_{j-1/2})-q^{+}(p_{j+1}+2p_{j+1/2})),$$

$$T_{j3}^{II}=(\varepsilon h/6)(r^{+}-r^{-})+(\varepsilon h/2)(q^{-}-q^{+})$$
$$-(h^{2}/12)(q^{-}(p_{j-1}+p_{j-1/2}/4)+q^{+}(p_{j+1}+p_{j+1/2}/4)),$$

$$R_{j4}^{II}(y)=T_{jr}^{II}+T_{jq}^{II}$$

$$T_{jr}^{II}=(\varepsilon h^{-2})r^{-}R_{3}(x_{j},x_{j-1},y)+(\varepsilon h^{-2})r^{+}R_{3}(x_{j},x_{j+1},y)$$

$$T_{jq}^{II}=(-q^{-}\varepsilon/6)(R_{1}(x_{j},x_{j-1},y'')+4R_{1}(x_{j},x_{j-1/2},y''))-$$
$$-(q^{+}\varepsilon/6)(R_{1}(x_{j},x_{j+1},y'')+4R_{1}(x_{j},x_{j+1/2},y''))-$$
$$-(q^{-}/6)(p_{j-1}R_{2}(x_{j},x_{j-1},y')+p_{j-1/2}R_{2}(x_{j},x_{j-1/2},y'))-$$
$$-(q^{+}/6)(p_{j+1}R_{2}(x_{j},x_{j+1},y')+p_{j+1/2}R_{2}(x_{j},x_{j+1/2},y')).$$

After some Taylor's expansions, in the case $h\leq\varepsilon$ we obtain:

(3.1) $$T_{j2}^{I}=-(h^{2}/6)p'(x_{j})+O(h^{3}/\varepsilon)p''(\beta_{2}),$$

and :

(3.2) $$T_{j2}^{II}=-(h^{2}/12)p'(x_{j})+O(h^{3}/\varepsilon)p''(\beta_{3}),$$

where

$$x_{j-1} < \beta_1, \beta_2 < x_{j+1}, \qquad x_{j-1/2} < \beta_3 < x_{j+1/2}.$$

If $p(x) = p = const$ we find :

$$T_{j3}^{I} = (-h^2/12)p + O(h^3/\varepsilon), \tag{3.3}$$

and

$$T_{j3}^{II} = (-h^2/24)p + O(h^3/\varepsilon). \tag{3.4}$$

Comparing T_{j2}^{I} with T_{j2}^{II}, and T_{j3}^{I} with T_{j3}^{II}, when $h \leq \varepsilon$, we conclude the *IEMW* scheme is twice as good as the *IIEMW* scheme. This fact indicate the mentioned linear combination (2.9). Numerical results presented in the first rows of Table1 and Table 2 illustrate this relation.

More interesting is the behaviour of the truncation errors when $\varepsilon < h$. In that case we find that the following esimates are valid.

$$\lim_{\varepsilon \to 0} T_{j2}^{II} = -(h^2/12)p'(x_j), \tag{3.5}$$

and

$$\lim_{\varepsilon \to 0} T_{j2}^{I} = 0, \quad \text{since} \quad |T_{j2}| \leq Mh\varepsilon. \tag{3.6}$$

From estimate (3.6) we find that for the IEMW scheme $\lim_{\varepsilon \to 0} \tau_j(y) = 0$ when $y \in P_2$, where P_2 is a set of polynomials of the degree less or equal to 2.

The behaviour of T_{j3} is very interesting for both schemes. Namely, when $p(x) = p = const$ we obtain

$$\lim_{\varepsilon \to 0} T_{j3}^{II} = 0, \quad \text{since} \quad |T_{j3}| \leq Mh\varepsilon, \tag{3.7}$$

and

$$\lim_{\varepsilon \to 0} T_{j3}^{I} = (h^2/24)p. \tag{3.8}$$

Now we can prove the following theorem:

Theorem 3.1. *Let in* (1.1) $d(x) \equiv 0$ *and* $y(x) \in C^6(I)$. *Let* u_j *be approximation to* $y(x_j)$ *obtained using scheme* (2.10),(2.11). *Then*

$$\max_{0 \leq j \leq n} |y(x_j) - u_j| \leq Mh^4/(\varepsilon^2 + h^2)$$

where M *is a constant independent of* ε *and* h.

Proof. The proof follows from the proof for the EMW scheme given in [1] and the facts that

$$(p(x_j) + 4p(x_{j\pm 1/2}) + p(x_{j\pm 1}))/6 = (p(x_j) + p(x_{j\pm 1}))/2 + R_1$$

and

$$(f(x_j) + 4f(x_{j\pm 1/2}) + f(x_{j\pm 1}))/6 = (f(x_j) + f(x_{j\pm 1}))/2 + R_2$$

where $|R_1|, |R_2| \leq Mh^2$. □

From (3.3) and (3.8) one can see that for the *IEMW* scheme T_{j3} changes the sign when ε goes to zero, for fixed h. As T_{j3} is a continous function of ε, at a certain point it becomes zero which may contribute to error decreasing for small ε.

The superriority of the *IIEMW* scheme when $\varepsilon << h$ results from the facts that $\lim_{\varepsilon \to 0} T_{j3}^{II} = 0$ and that the contribution of T_{j2}^{II} to the error is small if the function $p(x)$ is such a function that $p'(x)$ is small. The numerical results presented in the last rows of Table1 and Table2 support the given estimates. The adventage of the *IIEMW* scheme over the *IEMW* scheme can not be expected if the function $p(x)$ dasn't possess this property.

From the given analysis we canclude that which one of the schemes can be regarded adventageous depends on the behaviour of the functuion $p(x)$.

The good behaviour of schemes *IEMW* and *IIEMW*, for small ε, is lost in the linear combination (2.9). If we want to improve that, we may use the scheme:

(3.9) $$2 \cdot (1 - \lambda) \cdot \text{IEMW} \; - \lambda \cdot \text{IIEMW}.$$

When $\lambda = 1/2$ the scheme (3.9) obtains the form (2.9). When ε is very small we can take $\lambda = \varepsilon$ or $\lambda = 1$ in order to use the mentioned adventages of the schemes *IEMW* or *IIEMW*.

The numerical results presented in Table 4 indicate that the error estimate given in Theorem 1 holds for $d(x) \neq 0$.

4. Numerical Results

In this section we present the results of some numerical experiments using the *IEMW*, *IIEMW* schemes and the new scheme (2.9).

We denote by E_n the maximum of $|y(x_j) - u_j^n|$, $j = 0, ..., n$, by I_n the maximum of $|u_j^n - u_{2j}^{2n}|$, $j = 0, ..., n$, where u_j^{2n} and u_j^n denote approximate solutions at the mesh points for two successive values of n.

The order of convergence Ord, we define in the usual way:

$$Ord = \frac{\log E_n - \log E_{2n}}{\log 2},$$

or

$$Ord = \frac{\log I_n - \log I_{2n}}{\log 2},$$

when the exact solution is anknown.

Different values of $\varepsilon = 2^{-k}$ $k = 1, ..., 15$ and n are considered.

As numerical example we shall consider the boundary value problem ([12]):

Example 4.1.

$$\varepsilon y''(x) + (2 + 2\varepsilon(1 + x))/(1 + x)^2 y' = f(x, \varepsilon)$$

$$y(x) = \cos(\pi x/(1+x)) + (\exp(-1/\varepsilon) - \exp(-2x/(\varepsilon(1+x))))/(1 - \exp(-1/\varepsilon)).$$

The solution $y(x)$ determines $f(x, \varepsilon), \alpha_0$ and α_1. The derivatives of $f(x, \varepsilon)$ are bounded functions of ε.

The boundary value problem ([1]):

Example 4.2.

$$\varepsilon y''(x) + (x + 1)^3 y'(x) + 0.31(x + 1)^5 y(x) = -0.43 - 0.29x - 23x^2$$

$$y(0) = 2.7\ ,\ y(1) = 0.53.$$

Table 1 and Table 2 contain numerical order of the convergence Ord obtained using the IIEMW scheme and IEMW scheme, respectively. Table 3 and Table 4 presents the corresponding results obtained using the new scheme (2.9).

k	n							
	16	32	64	128	256	512	1024	
1	7.25(-4)	1.83(-4)	4.57(-5)	1.14(-5)	2.86(-6)	7.15(-7)	1.79(-7)	E_n
	2.47	2.67	2.82	2.88	2.89			Ord
2	3.52(-4)	8.92(-5)	2.24(-5)	5.61(-6)	1.40(-6)	3.51(-7)	8.77(-8)	E_n
	2.14	2.02	2.15	2.44	2.62			Ord
3	1.11(-3)	2.78(-4)	6.87(-5)	1.71(-5)	4.27(-6)	1.07(-6)	2.67(-7)	E_n
	2.01	2.05	2.29	2.51	2.65			Ord
4	2.17(-3)	5.21(-4)	1.30(-4)	3.25(-5)	8.12(-6)	2.03(-6)	5.07(-7)	E_n
	2.07	2.00	2.13	2.41	2.60			Ord
5	2.72(-3)	7.02(-4)	1.82(-4)	4.56(-5)	1.14(-5)	2.85(-6)	7.13(-7)	E_n
	1.94	1.95	2.00	2.19	2.46			Ord
6	2.22(-3)	7.95(-4)	2.15 (-3)	5.50(-5)	1.38(-5)	3.46(-6)	8.65(-7)	E_n
	1.41	1.86	1.96	2.00	2.21			Ord
7	1.47(-3)	6.36(-4)	2.15(-4)	5.92(-5)	1.53(-5)	3.86(-6)	9.68(-7)	E_n
	1.14	1.15	1.82	1.94	2.00			Ord
8	9.33(-4)	4.27(-4)	1.70(-4)	5.58(-5)	1.58(-5)	4.08(-6)	1.03(-6)	E_n
	1.11	1.24	1.52	1.78	1.94			Ord
9	6.01(-4)	2.81(-4)	1.15(-4)	4.41(-5)	1.44(-5)	4.09(-6)	1.06(-6)	E_n
	1.14	1.28	1.30	1.51	1.78			Ord
10	4.33(-4)	1.94(-4)	7.70(-5)	2.99(-5)	1.12(-5)	3.69(-6)	1.04(-6)	E_n
	1.20	1.37	1.36	1.33	1.51			Ord
11	3.41(-4)	1.47(-4)	5.45(-5)	2.02(-5)	7.62(-6)	2.83(-6)	9.33(-7)	E_n
	1.25	1.48	1.48	1.40	1.35			Ord
12	2.94(-4)	1.22(-4)	4.23 (-5)	1.44(-5)	5.16(-6)	1.92(-6)	7.11(-7)	E_n
	1.29	1.58	1.61	1.53	1.42			Ord
13	2.70(-4)	1.10(-4)	3.60(-5)	1.13(-5)	3.72(-6)	1.30(-6)	4.83(-7)	E_n
	1.32	1.64	1.72	1.67	1.56			Ord
14	2.58(-4)	1.03(-4)	3.28(-5)	9.75(-6)	2.94(-6)	9.43(-7)	3.28(-7)	E_n
	1.33	1.66	1.78	1.78	1.70			Ord
15	2.52(-4)	1.00(-4)	3.12(-5)	8.93(-6)	2.53(-6)	7.47(-7)	2.37(-7)	E_n
	1.34	1.69	1.82	1.85	1.82			Ord

Table 1. Example 1, the *IIEMW* scheme

k	n							
	16	32	64	128	256	512	1024	
1	3.65(-4)	9.15(-5)	2.29(-5)	5.72(-6)	1.43(-6)	3.57(-7)	8.94(-8)	E_n
	2.49	2.69	2.82	2.88	2.89			Ord
2	1.77(-4)	4.47(-5)	1.12(-5)	2.80(-6)	7.01(-7)	1.75(-7)	4.38(-8)	E_n
	2.15	2.03	2.15	2.44	2.62			Ord
3	5.67(-4)	1.40(-4)	3.44(-5)	8.56(-6)	2.14(-6)	5.34(-7)	1.34(-7)	E_n
	2.04	2.06	2.30	2.51	2.65			Ord
4	1.12(-3)	2.63(-4)	6.51(-5)	1.62(-5)	4.06(-6)	1.01(-6)	2.54(-7)	E_n
	2.10	2.02	2.13	2.41	2.60			Ord
5	1.27(-3)	3.46(-4)	9.08(-5)	2.28(-5)	5.70(-6)	1.42(-6)	3.56(-7)	E_n
	1.85	1.92	2.00	2.19	2.46			Ord
6	7.04(-4)	3.51(-4)	1.04 (-4)	2.73(-5)	6.89(-6)	1.73(-6)	4.33(-7)	E_n
	0.72	1.68	1.92	1.98	2.20			Ord
7	4.99(-5)	1.69(-4)	9.15(-5)	2.84(-5)	7.58(-6)	1.93(-6)	4.83(-7)	E_n
	0.94	0.41	1.59	1.89	1.98			Ord
8	5.36(-4)	3.71(-5)	4.10(-5)	2.33(-5)	7.52(-6)	2.02(-6)	5.14(-7)	E_n
	2.76	2.06	0.18	1.53	1.87			Ord
9	8.51(-4)	1.81(-4)	1.42(-5)	1.00(-5)	5.92(-6)	1.94(-6)	5.23(-7)	E_n
	2.06	2.83	2.54	0.04	1.49			Ord
10	1.02(-3)	2.67(-4)	5.21(-5)	4.21(-6)	2.48(-6)	1.50(-6)	4.94(-7)	E_n
	1.87	2.19	2.86	2.77	-0.06			Ord
11	1.11(-3)	3.14(-4)	7.44(-5)	1.39(-5)	1.14(-6)	6.16(-7)	3.79(-7)	E_n
	1.79	2.01	2.26	2.88	2.89			Ord
12	1.16(-3)	3.38(-4)	8.65 (-5)	1.96(-5)	3.59(-6)	2.95(-7)	1.53(-7)	E_n
	1.76	1.93	2.07	2.29	2.88			Ord
13	1.18(-3)	3.50(-4)	9.28(-5)	2.27(-5)	5.03(-6)	9.13(-7)	7.52(-8)	E_n
	1.75	1.90	2.00	2.11	2.30			Ord
14	1.19(-3)	3.56(-4)	9.60(-5)	2.43(-5)	5.81(-6)	1.27(-6)	2.30(-7)	E_n
	1.74	1.89	1.98	2.03	2.12			Ord
15	1.20(-3)	3.60(-4)	9.76(-5)	2.51(-5)	6.21(-6)	1.47(-6)	3.21(-7)	E_n
	1.73	1.88	1.95	2.00	2.05			Ord

Table 2. Example 1, the *IEMW* scheme

k	n							
	16	32	64	128	256	512	1024	
1	8.17(−6)	5.11 (−7)	3.19(−8)	1.99(−9)	1.25(−10)	7.77(−12)	2.84(−13)	E_n
	4.05	4.27	4.47	4.61	4.68			Ord
2	2.99(−6)	1.75(−7)	1.08(−8)	6.72(−10)	4.19(−11)	2.72(−12)	5.22(−13)	E_n
	3.76	3.87	3.99	4.16	4.43			Ord
3	3.51(−5)	2.33(−6)	1.48(−7)	9.35(−9)	5.85(−10)	3.66(−11)	2.07(−12)	E_n
	3.91	3.98	4.00	4.25	4.48			Ord
4	6.20(−5)	6.68(−6)	4.39(−7)	2.78(−8)	1.75(−9)	1.09(−10)	6.85(−12)	E_n
	3.20	3.93	3.98	3.99	4.20			Ord
5	1.71(−4)	1.08(−5)	7.34(−7)	5.35(−8)	3.51(−9)	2.22(−10)	1.38(−11)	E_n
	3.99	3.88	3.77	3.93	3.96			Ord
6	8.12(−4)	9.43(−5)	6.78(−6)	4.50(−7)	2.84(−8)	1.78(−9)	1.11(−10)	E_n
	3.07	3.79	3.91	4.10	4.95			Ord
7	9.92(−4)	2.98(−4)	3.19(−5)	2.46(−6)	1.63(−7)	1.03(−8)	6.49(−10)	E_n
	2.19	3.18	3.68	3.91	4.04			Ord
8	4.44(−4)	2.72(−4)	8.86(−5)	9.44(−6)	7.38(−7)	4.89(−8)	3.10(−9)	E_n
	-0.65	2.14	3.18	3.66	3.91			Ord
9	1.11(−4)	1.23(−4)	7.10(−5)	2.41(−5)	2.60(−6)	2.01(−7)	1.34(−8)	E_n
	0.19	-0.67	2.12	3.16	3.68			Ord
10	1.46(−4)	3.35(−5)	3.21(−5)	1.80(−5)	6.27(−6)	6.82(−7)	5.29(−8)	E_n
	3.08	-0.03	-0.69	2.11	3.15			Ord
11	1.92(−4)	3.86(−5)	9.00(−6)	8.13(−6)	4.53(−6)	1.60(−6)	1.75(−7)	E_n
	2.28	3.64	-0.52	-0.72	2.10			Ord
12	2.30(−4)	5.06(−5)	1.00(−6)	2.28(−6)	2.03(−6)	1.13(−6)	4.04(−7)	E_n
	2.11	2.40	4.36	-1.26	-0.74			Ord
13	2.54(−4)	5.98(−5)	1.30(−5)	2.54(−6)	5.61(−7)	5.05(−7)	2.82(−7)	E_n
	2.05	2.15	2.51	4.86	-1.81			Ord
14	2.70(−4)	6.55(−5)	1.53(−5)	3.32(−6)	6.42(−7)	1.36(−7)	1.25(−7)	E_n
	2.02	2.07	2.20	2.57	5.17			Ord
15	2.80(−4)	6.92(−5)	1.67(−5)	3.88(−6)	8.40(−7)	1.62(−7)	3.30(−8)	E_n
	2.00	2.03	2.08	2.23	2.59			Ord

Table 3. Example 1, New scheme (2.9)

k	n					
	16	32	64	128	256	
1	3.86	4.27	4.49	4.62	4.00	Ord
2	4.15	4.46	4.60	4.70	4.36	Ord
3	3.93	4.42	4.63	4.72	4.75	Ord
4	3.35	4.10	4.51	4.70	4.79	Ord
5	2.72	3.49	4.15	4.55	4.74	Ord
6	2.39	2.91	3.54	4.15	4.55	Ord
7	2.28	2.57	3.03	3.55	4.13	Ord
8	1.82	2.42	2.72	3.07	3.54	Ord
9	1.66	2.07	2.48	2.77	3.10	Ord
10	1.59	1.91	2.18	2.51	2.80	Ord
11	1.56	1.85	2.02	2.24	2.53	Ord
12	1.55	1.82	1.96	2.08	2.26	Ord
13	1.54	1.80	1.93	2.01	2.10	Ord
14	1.54	1.79	1.91	1.99	2.04	Ord
15	1.54	1.79	1.90	1.96	2.00	Ord

Table 4. Example 2, New scheme (2.9)

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