Novi Sad J. Math.
Vol. 28, No. 2, 1998, 33-42

SOME UNIFORMLY CONVERGENT SCHEMES ON SHISHKIN MESH

Vanja Vukoslavčević
Faculty of Sciences
University of Podgorica
Cetinjska b.b., 81000 Podgorica
Yugoslavia

Katarina Surla, Sanja Rapajić
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad
Yugoslavia

Abstract

The uniformly convergent spline difference scheme for self-adjoint problem is derived. The convergence $O(n^{-2}\ln^2 n)$ on Shishkin mesh is achived.

AMS Mathematics Subject Classification (1991): 65L10
Key words and phrases: Spline collocation, difference scheme, singular perturbation problem, uniform convergence

1. Introduction

Consider the problem

$$(1)\qquad \begin{cases} Ly \equiv -\varepsilon y'' + p(x)y = f(x), & x \in I = (0,1), \\ y(0) = 0, \quad y(1) = 0, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a small parameter. The function p and f are given and we assume

$$p, f \in C^2(I), \quad p(x) \geq \beta^2 > 0.$$

It is known that the problem (1) has a unique solytion y, which in general displays boundary layers at $x = 0$ and $x = 1$. Under the above assumptions the exact solution has the form [1]:

$$y(x) = v(x) + g(x),$$

where

$$|g^{(j)}(x)| \leq M, \quad j = 1, 2, 3, 4,$$

$$|v^{(j)}(x)| \leq M\varepsilon^{-j/2}(e^{-x\frac{b}{\sqrt{\varepsilon}}} + e^{(x-1)\frac{b}{\sqrt{\varepsilon}}}) \quad j = 1, 2, 3, 4,$$

$b = \min(\beta, 1)$.

Throughout the paper M denotes any positive constant that may take different values in different formulas, but that are allways independent of ε and discretization mesh.

In [10], the family of difference schemes for problem (1) via cubic spline on uniform mesh are derived. By using the artifical viscosity the uniform convergence at the mesh point is obtained. In this paper we derive the anologous family on the non-uniform mesh. The member of the family which coresponds to Hegarty scheme [4], is analysed. By using the nouniform mesh of Shishkin type the uniform convergence is achieved without the exponential fitting.

2. Derivation of the scheme

The solution of the problem (1) is sought in the form of the cubic spline $u(x) \in C^1(I)$ on non-uniform mesh

$$0 = x_0 < x_1 < .. < x_{i_0} < .. < x_{n-i_0} < .. < x_n = 1 \tag{2}$$

The function $u(x)$ on each interval $I_j = [x_j, x_{j+1}]$ has the form:

$$u_j(x) = u_j + (x - x_j)u_j' + \frac{(x - x_j)^2}{2}u_j'' + \frac{(x - x_j)^3}{3!}u_j'''.$$

From the requirements

$$-\varepsilon u_j'' + p_j^- u_j = f_j^-,$$

$$-\varepsilon u''_{j+1} + p_j^+ u_{j+1} = f_j^+,$$

where p_j^-, p_j^+ are approksimate functions for p on I_{j-1} and I_j respectively and analogously for f, and $u(x) \in C^1(I)$ we obtain the scheme

$$\text{(3)} \qquad \begin{cases} Ru_j = Qf_j, \quad j = 1(1)n-1 \\ u_0 = 0, \quad u_n = 0, \end{cases}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1},$$

$$Qf_j = q_j^- f_j^- + q_j^+ f_{j+1}^+,$$

$$r_j^- = G(p_j^-, h_j) - \frac{1}{h_j}, \quad r_j^+ = G(p_j^+, h_{j+1}) - \frac{1}{h_{j+1}},$$

$$r_j^c = 2G(p_j^-, h_j) + \frac{1}{h_j} + 2G(p_j^+, h_{j+1}) + \frac{1}{h_{j+1}},$$

$$\text{(4)} \qquad G(p,h) = \frac{ph}{6\varepsilon}, \quad h_j = x_j - x_{j-1},$$

$$q_j^- = \frac{h_j}{2\varepsilon}, \quad q_j^+ = \frac{h_{j+1}}{2\varepsilon}.$$

If we use $p_j^- = p_j^+ = p_j$ and $f_j^- = f_j^+ = f_j$ the scheme (3), (4) reducet to central difference scheme:

$$\text{(5)} \qquad \begin{cases} r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = f_j, \\ u_0 = 0, \quad u_n = 0, \quad j = 1(1)n-1, \end{cases}$$

where

$$r_j^- = -\frac{2\varepsilon}{h_j(h_j + h_{j+1})}, \quad r_j^+ = -\frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})},$$

$$r_j^c = \frac{2\varepsilon}{h_j h_{j+1}} + p_j.$$

When we use the approximate function in the form

$$p_j^- = p_j + \frac{p_{j-1} - p_j}{x_{j-1} - x_j}(x - x_j), \quad p_j^+ = p_{j+1} + \frac{p_j - p_{j+1}}{x_j - x_{j+1}}(x - x_{j+1}),$$

$$f_j^- = f_j + \frac{f_{j-1} - f_j}{x_{j-1} - x_j}(x - x_j), \qquad f_j^+ = f_{j+1} + \frac{f_j - f_{j+1}}{x_j - x_{j+1}}(x - x_{j+1}),$$

we obtain the scheme

$$\text{(6)} \qquad \begin{cases} r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q_j^- f_j + q_j^c f_j + q_j^+ f_{j+1}, \\ u_0 = 0, \quad u_n = 0, \quad j = 1(1)n-1, \end{cases}$$

where

$$\text{(7)} \qquad \begin{aligned} r_j^- &= -(1 - \frac{h_j^2 p_{j-1}}{6\varepsilon})\frac{1}{h_j}, \qquad r_j^+ = -(1 - \frac{h_{j+1}^2 p_{j+1}}{6\varepsilon})\frac{1}{h_{j+1}}, \\ r_j^c &= (1 + \frac{h_j^2 p_j}{3\varepsilon})\frac{1}{h_j} + (1 + \frac{h_{j+1}^2 p_j}{3\varepsilon})\frac{1}{h_{j+1}}, \\ q_j^- &= \frac{h_j}{6\varepsilon}, \qquad q_j^c = \frac{h_j}{3\varepsilon} + \frac{h_{j+1}}{3\varepsilon}, \qquad q_j^+ = \frac{h_{j+1}}{6\varepsilon}. \end{aligned}$$

For

$$p_j^- = \frac{p_{j-1} + p_j}{2}, \qquad p_j^+ = \frac{p_j + p_{j+1}}{2},$$

and analogously for f_j^-, f_j^+, difference scheme (3), (4) has the form:

$$\text{(8)} \qquad \begin{cases} r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q_j^- f_j + q_j^c f_j + q_j^+ f_{j+1}, \\ u_0 = 0, \quad u_n = 0, \quad j = 1(1)n-1, \end{cases}$$

where

$$\text{(9)} \qquad \begin{aligned} r_j^- &= G(p_{j-1} + p_j, h_j)/2 - \frac{1}{h_j}, \qquad r_j^+ = G(p_{j+1} + p_j, h_{j+1})/2 - \frac{1}{h_{j+1}}, \\ r_j^c &= G(p_{j-1} + p_j, h_j) + \frac{1}{h_j} + G(p_{j+1} + p_j, h_{j+1}) + \frac{1}{h_{j+1}}, \\ q_j^- &= \frac{h_j}{4\varepsilon}, \qquad q_j^c = \frac{h_j}{4\varepsilon} + \frac{h_{j+1}}{4\varepsilon}, \qquad q_j^+ = \frac{h_{j+1}}{4\varepsilon}. \end{aligned}$$

We observe mentioned schemes on Shishkin mesh.

3. Shishkin mesh

Shishkin mesh is piecewise equidistant. If we devide the interval $[0,1]$ into the three subintervals

$$[0,\sigma], \quad [\sigma, 1-\sigma] \text{ and } [1-\sigma, 1],$$

we construct a equidistant meshes on each of these subintervals, with $1+n/4$ points in each of $[0,\sigma]$ and $[1-\sigma,1]$ and $1+n/2$ points in $[\sigma, 1-\sigma]$, where n is given positive inreger and n is divisible by 4.

The parameters σ and b are defined by

$$b = \min\{\beta, 1\}, \quad \sigma = \min\{1/4, 4b^{-1}\sqrt{\varepsilon}\ln n\} \tag{10}$$

with $i_0 = n/4$, $\quad x_{i_0} = \sigma$, $\quad x_{n-i_0} = 1-\sigma$ and

$$h_i = 4\sigma n^{-1}, \quad i = 1,..,i_0, n-i_0+1,..n,$$

$$h_i = 2(1-2\sigma)n^{-1}, \quad i = i_0+1,..,n-i_0,$$

and $h_i = x_i - x_{i-1}, i = 1(1)n$.

The scheme (5) on Shishkin mesh is considered in [..] and the allmost second order of the convergence is obtained.

The scheme (6), (7) on Shishkin mesh is analysed in [..] and also the allmost second order of the convergence is obtained.

In this paper we consider the scheme (8), (9) on the Shishkin mesh .

4. Convergence result

We will estimate the values $z_j = y(x_j) - u_j$. In this purpose we use the inequalities

$$|z_j| \le \|A^{-1}\| \max_j |\tau_j(y)| \tag{11}$$

where A is the matrix of the scheme (8), (9) and $\tau_j(y)$ is the truncation error defined by

$$\tau_j(y) = Ry_j - (QLy)_j = Rz_j.$$

Function $z(x) = y(x) - u(x)$ satisfied the equations

$$(12)\quad \begin{cases} z_j = z_{j-1} + h_j z_j' + h_j^2 z_j''/2 + h_j^3 z_j''' + R_{0,j}, \\ z_j' = z_{j-1}' + h_j z_j'' + h_j^2 z_j'''/2 + R_{1,j}, \\ z_j'' = z_{j-1}'' + h_j z_j''' + R_{2,j}, \\ R_{k,j} = h_j^{4-k} z^{IV}(\nu_{j_k}), \qquad x_{j-1} \le \nu_{j_k} \le x_j, \\ -\varepsilon z_{j-1}'' + \bar{p}_j z_{j-1} = f_{j-1} - \bar{f}_{j-1} + (\bar{p}_{j-1} - p_{j-1}) z_{j-1}, \end{cases}$$

where $\bar{p}_{j-1} = \frac{p_j + p_{j-1}}{2}$ and analogously for $\bar{f}_{j-1}$, $j = 1(1)n$.

From the equations (12) using the fact $z(x) \in C^1(I)$ we obtain that

$$(13)\qquad \tau_j(y) = Rz_j = -\frac{\varphi_j}{\gamma_j} - \frac{a_j \varphi_{j-1}}{\gamma_{j-1}} + G_{j-1},$$

where

$$\varphi_j = h_{j+1}[Q_{1_j}(-\frac{h_{j+1}}{2\varepsilon} - \frac{h_{j+1}^4}{12\varepsilon} p_j \alpha_j + \frac{h_{j+1}^2}{6}\alpha_j) - \frac{h_{j+1}^2}{6} Q_j \alpha_j) - \frac{R_{0,j}}{h_{j+1}}],$$

$$Q_{1_j} = (\bar{p}_j - p_j) y_j + f_j - \bar{f}_j = \varepsilon y_j'' - \varepsilon u_j'',$$

$$Q_{2_j} - Q_{1_j} = \frac{\varepsilon h_{j+1}^2}{2} y^{IV}(\nu_{j_1}) - \bar{p}_j \frac{h_{j+1}^4}{24} y^{IV}(\nu_{j_4}),$$

$$\alpha_j = \frac{6}{(-6\varepsilon + h_{j+1}^2 \bar{p}_j) h_{j+1}},$$

$$\gamma_j = -\frac{6\varepsilon h_{j+1}}{-6\varepsilon + h_{j+1}^2 \bar{p}_j},$$

$$a_j = -\frac{6\varepsilon + 2h_{j+1}^2 \bar{p}_j}{-6\varepsilon + h_{j+1}^2 \bar{p}_j},$$

$$G_{j-1} = -\frac{h_j Q_{1_{j-1}}}{\varepsilon} + \frac{h_j^4}{4\varepsilon} \bar{p}_{j-1} Q_{1_{j-1}} \alpha_{j-1} + \alpha_{j-1} \frac{h_j^2}{2} (Q_{1_{j-1}} - Q_{2_{j-1}}) + R_{1,j-1}.$$

Becouse of linearity we have

$$\tau_j(y) = \tau_j(g) + \tau_j(v).$$

It is convinient to consider scheme (8), (9) the form which we obtain by multipling the each equation by $\mu_j = \frac{\varepsilon}{h_j+h_{j+1}}$. The new matrix we denote by $\bar{A}$, $\bar{\tau}_j(y)$ is corresponding truncation error and $R_j^- = \mu_j r_j^-$ and so on. From (13) for $i = 1(1)i_0 - 1$ and for $i = n - i_0 + 1(1)n - 1$ we have that

$$|\bar{\tau}_j(y) \leq \mu_j M \frac{n^{-3}\ln^3 n}{\sqrt{\varepsilon}},$$

For $i = i_0 + 1(1)n - i_0 - 1$ we have

$$|\bar{\tau}_j(y) \leq \mu_j M n^{-3}.$$

At the transition points we consider separatly $\bar{\tau}_j(g)$ and $\bar{\tau}_j(v)$. Using the estimate $|Q_{1_j}(g)| \leq M\varepsilon$ we obtain for $j = i_0$

$$|\bar{\tau}_j(g) \leq \mu_j M (h_{j+1} + \frac{h_{j+1}^3}{\varepsilon}).$$

For $\bar{\tau}_j(v)$, we use the form

$$\bar{\tau}_j(v) = \mu_j[v_{j-1}(r_j^- - p_{j-1}q_j^-) + +v_j(-r_j^c - p_j q_j^c) +$$

$$+v_{j+1}(r_j^+ - p_{j+1}q_j^+) - \varepsilon(q_j^- v_{j-1}'' + q_j^c v_j'' + q_j^+ v_{j+1}'')].$$

Thus,

$$|\bar{\tau}_{i_0}(v)| \leq M\mu_{i_0}(n^{-5}\ln n + \frac{n^{-4}}{\sqrt{\varepsilon}\ln n} + \frac{n^{-5}}{\varepsilon} + n^{-3}).$$

Further

$$\Delta_j = R_j^c - |R_j^-| - |R_j^+| \geq M$$

and

$$\|\bar{A}^{-1}\| \leq M$$

and from (11) we have

$$|z_i| \leq M(\varepsilon + n^{-2}\ln^2 n)$$

which for small ε give us near the second order of the convergence. Thus, the folloving theorem is proved

Theorem 1 *Let $p, f \in C^2(I)$, $p(x) \geq \beta^2 > 0$. Let u_j be the solution of the sistem (8), (9). Then*

$$|y(x_i) - u_i| \leq M(\varepsilon + n^{-2}\ln^2 n).$$

Remark. The same results we can obtain if we choise

$$\sigma = \min(1/4, 2b^{-1}\sqrt{\varepsilon}\ln n).$$

5. Numerical results

In this section we present results of some numerical experiments using the schemes described in previous theorem. Our examples are taken from [1].

Example 1

$$-\varepsilon y'' + (1 + x(1-x))y = f(x), \qquad u(0) = 0, \qquad u(1) = 0.$$

The exact solution has the form

$$y(x) = 1 - (1-x)e^{\frac{-x}{\sqrt{\varepsilon}}} - xe^{\frac{x-1}{\sqrt{\varepsilon}}}$$

We denote by E_n the maximum of $|y(x_j) - u_j|, j = 0(1)n$. Here $[u_0, u_1, \ldots, u_n]^\top$ is corresponding numerical solution obtained by using scheme (8),(9). Assuming converrgence of order $(n^{-1}\ln n)^r$ for some r, we estimate the classical convergence rate r from

$$R_\varepsilon^n = \frac{\ln \tilde{E}_\varepsilon^{2n} - \ln \tilde{E}_\varepsilon^n}{\ln(\frac{2k}{k+1})}, \; n = 2^k, \; k = 4, 5, 6, 7, 8,$$

where

$$\tilde{E}_\varepsilon^n = \max_{0 \le i \le n} |(u^n)_i - (\tilde{u}^{2n})_{2i}|$$

and $\tilde{u}^n \in R^{n+1}$ be a solution on a Shishkin mesh with the parameter σ altered slightly to

$$\sigma = \min\{1/4, 4b^{-1}\sqrt{\varepsilon}\ln(n/2)\}.$$

Different values of $\varepsilon = 2^p$ and n are considered.

p	n							
	16	32	64	128	256	512	1024	
4	2.81(−3)	6.99(−4)	1.74(−4)	4.36(−5)	1.09(−5)	2.72(−6)	6.81(−7)	E_n
	2.96	2.72	2.56	2.47	2.41			Ord
6	7.06(−3)	1.73(−3)	4.34(−4)	1.08(−4)	2.71(−5)	6.78(−6)	1.69(−6)	E_n
	3.00	2.70	2.57	2.48	2.40			Ord
8	2.41(−2)	5.58(−3)	1.37(−3)	3.41(−4)	8.54(−5)	2.13(−5)	5.33(−6)	E_n
	3.14	2.77	2.56	2.47	2.40			Ord
10	7.48(−2)	2.06(−2)	4.78(−3)	1.17(−3)	2.92(−4)	7.31(−5)	1.82(−5)	E_n
	2.86	2.89	2.61	2.48	2.40			Ord
12	1.09(−1)	5.47(−2)	1.87(−2)	4.36(−3)	1.07(−3)	2.66(−4)	6.66(−5)	E_n
	1.62	2.16	2.73	2.51	2.41			Ord
14	1.05(−1)	5.24(−2)	1.94(−2)	6.01(−3)	1.96(−3)	6.17(−4)	1.90(−4)	E_n
	1.63	2.01	2.20	2.01	2.00			Ord
16	1.04(−1)	5.11(−2)	1.88(−2)	5.86(−3)	1.91(−3)	6.01(−4)	1.85(−4)	E_n
	1.64	2.02	2.19	2.02	2.00			Ord
18	1.03(−1)	5.05(−2)	1.86(−2)	5.78(−3)	1.88(−3)	5.92(−4)	1.82(−4)	E_n
	1.65	2.02	2.19	2.02	2.00			Ord
20	1.02(−1)	5.02(−2)	1.84(−2)	5.74(−3)	1.87(−3)	5.88(−4)	1.81(−4)	E_n
	1.65	2.02	2.19	2.02	2.00			Ord

Table 1. Scheme (8),(9).

References

[1] Doolan, E.P., Miller, J.J.H., Schilders, W.H.A., Uniform Numerical Methods for Problems with Initial and Boundary Layers. Dublin, Boole Press 1980. Press, Dublin 1980, 301-305.

[2] Hemker, P.W., A Numerical Study of Stiff Two-Point Boundary Problems. Amsterdam, Mathematical Centre 1977.

[3] Hess, W., Schmidt, W.J., Convexity preserving interpolation with exponential splines, Computing 36 (1986) 335-342.

[4] Hegarty, A.f.,Miller, J.J.H., O'Riordan, E., Uniform second order difference scheme for singular perturbation problem, boundary and interior layers-computational and asymptotic methods, ed. J.J.H. Miller, Boole Press, Dublin (1980) 301-305.

[5] O'Riordan, E., Finite Element Methods for Singularly Perturbed Problems. Dublin, Trinity College 1982 (Ph. D. thesis).

[6] Surla, K., Herceg, D., Cvetković, Lj., A family of exponentialy spline difference schemes, Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat., 19,2 (1991) 12–23.

[7] Surla, K.,Vukoslavčević, V., A spline difference scheme for boundary value problems with small parameter, Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat. 20,2(1995) 159-167.

[8] Vukoslavčevi c, V., Metoda konačnih elemenata u rešavanju singularno perturbovanih problema. Magistarski rad. Novi Sad 1994.

[9] Surla, K., Vukoslavčević, V., A spline difference scheme for boundary value problems with small parameter, Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat. 20,2 (1995) 159-167.

[10] Uzelac, Z., Spline Difference Scheme for Singular Perturbation Problemem, PhD. Novi Sad 1989.

Received by the editors May 1, 1996.