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MONOTONE ITERATIONS FOR SPECTRAL APPROXIMATION OF NONLINEAR LAYER PROBLEMS

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Abstract

In this paper we shall consider certain kinds of singularly perturbed problems described by quasilinear differential equation of second order with small parameter multiplying the highest derivative, and the appropriate boundary conditions, so that the solution displayes boundary layers. The character of the layer is determined by the use of asymptotic behavior of the exact solution out of the layer, where the exact solution is approximated by the solution of the reduced problem. *The ressemblance function* for the given problem is determined and used for the domain decomposition, so that the standard spectral methods can be applied inside the layer. The spectral approximation for the layer solution upon the layer subinterval is constructed using monotone iterations. The layer subinterval is determined through the numerical layer length which depends on the perturbation parameter and the degree of the chosen truncated orthogonal series used for the spectral approximation.

The error estimate is provided by the use of asymptotic behavior of the exact solution at the endpoints of the layer subintervals using the principle of inverse monotonicity.

The numerical example is included, showing the high accuracy of the presented method even when a small number of terms is used in the truncated orthogonal series, which is the result of the appropriatelly determined layer subinterval. The results are tested according to the Chebyshev orthogonal basis.

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1. Introduction

We shall consider a singularly perturbed problem

$$L_\varepsilon y \equiv -\varepsilon^2 y''(x) + b(x,y) = 0 \quad 0 \le x \le 1, \tag{1}$$

$$y(0) = 0, \quad y(1) = 0, \tag{2}$$

where $\varepsilon \in \mathbf{R}$, $0 < \varepsilon << 1$ is a small parameter.

The problem (1),(2) represents mathematical model of the large number of phenomena in sciences such as conduction and diffusion in fluid dynamics, theory of semiconductors in electronics and catalization processes in chemistry and biology. It is of the great interest to describe the behavior of the exact solution of these problems, especialy inside the layers. Under certain assumptions, the solution of this problem can display boundary layers, where the values change extremly rapidly.

Lately, quite a number of authors have tried to solve this kind of problems numericaly. They have developed two different aproaches in the construction of the approximate solution: discrete methods and continous methods.

Classical numerical discretization fails when applied to the layer problems and that difficulty is usualy overcome by the construction of special grid functions or special discretization schemas.

The authors who use the continous methods mostly use spline technique or finite element methods combined by the relaxation of the given problem.

In this paper a continous method, based on the use of spectral approximation will be presented.

In the first part of the paper we shall analyse the assumptions for the existence of the solution of the given boundary value problem and the asymptotic behavior of the solution. In the second part we shall transform the original problem addapting it to the application of standard spectral technique. In the third part we shall determine the appropriate domain decomposition introducing *numerical layer length* constructed by the special procedure which is addapted to the character of the chosen approximation. In the fourth part we shall apply monotone iterations to construct the sequence of spectral approximations which tends to the exact solution of the problem. In the fifth part we shall estimate the error, and in the sixth part we shall give a numerical example.

2. Existence and the behavior of the solution

The problem (1),(2) represents the so-called *selfadjoint problem*, which is the special case of stiff quasilinear boundary problem

$$T_\varepsilon y \equiv -\varepsilon^2 y''(x) + a(x,y)y'(x) + b(x,y) = 0, \quad 0 \le x \le 1,$$

$$Ry \equiv (y(0), y(1)) = (A, B),$$

where $\varepsilon > 0$ is a small parameter, $A, B \in \mathbf{R}$, and $a(x,y)$, $\frac{\partial a}{\partial y}$, $b(x,y)$ and $\frac{\partial b}{\partial y}$ are functions from the space $\mathbf{C}([0,1] \times \mathbf{R})$.

In [4] it was shown that if

$$a(x,y) = a(x), \qquad \frac{\partial b(x,y)}{\partial y} \ge 0,$$

the observed quasilinear problem has the unique solution $y_\varepsilon \in \mathbf{C}^2[0,1]$ and the operator (T_ε, R) is inverse monotone in the following sence

$$T_\varepsilon u \le T_\varepsilon v, \quad Ru \le Rv \quad \Rightarrow \quad u \le v \text{ for all } u, v \in \mathrm{C}^2[0,1].$$

The problem (1),(2) is obtained from the above quasilinear problem for $a(x,y) \equiv 0$ and homogenious boundary conditions $(A, B) = (0, 0)$. Thus, the existence and the uniqness of the solution, and the inverse monotonicity are obtained if the following assumption holds:

(3) $b_y(x,y) \ge \beta^2 > 0,\ \beta \in \mathbf{R},\ b(x,y) \in \mathbf{C}^2([0,1] \times \mathbf{R}),\ (x,y) \in [0,1] \times \mathbf{R}.$

The exact solution $y_\varepsilon(x)$ is from the space $\mathbf{C}^4[0,1]$ and its asymptotic behavior is given by

$$y_\varepsilon(x) \approx z(x) + M_0 e^{\frac{-\beta x}{\varepsilon}} + M_1 e^{\frac{-\beta(1-x)}{\varepsilon}} + M_2 \varepsilon^2, \tag{4}$$

where $z(x)$ represents the solution of the reduced problem $b(x,z)=0$ and

$$M_0 \geq |z(0)|\ ,\ M_1 \geq |z(1)|\ ,\ M_2 \geq \frac{z''(x)}{\beta^2}\ ,\ x \in [0,1].$$

We can se that, in general, the exact solution displays two boundary layers of order $O(\varepsilon)$.

3. Transformation of the problem

As the boundary layers are of order $O(\varepsilon)$, we shall approximate the exact solution $y_\varepsilon(x)$ by

$$u(x) = \begin{cases} u_l(x) & x \in [0, c\varepsilon] \\ z(x) & x \in [c\varepsilon, 1-c\varepsilon] \\ u_r(x) & x \in [1-c\varepsilon, 1] \end{cases}\ , \tag{5}$$

where $c > 0$ denotes a constant which choice will be discussed in the next section.

The function $u_l(x)$ represents *the left layer solution* and it can be determined as the solution of the problem

$$L_\varepsilon u_l(x) = 0\ ,\ u_l(0) = 0\ ,\ u_l(c\varepsilon) = z(c\varepsilon). \tag{6}$$

The function $u_r(x)$ represents *the right layer solution* and it can be determined as the solution of the problem

$$L_\varepsilon u_r(x) = 0\ ,\ u_r(0) = z(1-c\varepsilon)\ ,\ u_r(1) = 0. \tag{7}$$

4. Numerical layer length

We shall carry out the procedure for constructing the approximation only for the left layer solution, and the investigation for the right layer solution is the same.

The value c, which determines the division points $c\varepsilon$ and $1-c\varepsilon$ can be determined by the similar technique as in the linear case, for which one can see [1]. In that purpose we introduce thee following definition:

Definition 1. *A sum of the reduced solution and a function $p_m(x) \in \mathbf{C}^2[0, c\varepsilon]$ is called a resemblance function for the problem (6) if*

1. *it satisfies the boundary conditions in (6),*
2. *$x = c\varepsilon$ is the stationary point for $p_m(x)$,*
3. *$p_m(x)$ is concave for $z(0) < 0$ and convex for $z(0) > 0$.*

Now, it is easy to prove the following lemma:

Lemma 1. *The function*

$$(8)\quad r(x) = z(x) + p_m(x) = z(x) - z(0)\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^m, \quad m \in \mathbf{N}\,,\ m \geq 2.$$

is a resemblance function for the problem (6).

Proof. We have to verify the conditions from Defefinition 1.

1. $r(0) = z(0) - z(0)\left(\frac{c\varepsilon - 0}{c\varepsilon}\right)^m = 0$
   and
   $r(c\varepsilon) = z(c\varepsilon) - z(0)\left(\frac{c\varepsilon - c\varepsilon}{c\varepsilon}\right)^m = z(c\varepsilon)$.
2. $p'_m(x) = \frac{mz(0)}{c\varepsilon}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-1} = 0$ only for $x = c\varepsilon$.
3. $p''_m(x) = -\frac{m(m-1)z(0)}{c^2\varepsilon^2}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-2}$, so that $\text{sgn} p''_m(x) = -\,\text{sgn} z(0)$, which means that $p_m(x)$ is concave for $z(0) < 0$ and convex for $z(0) > 0$.

The resemblance function enables us to determine the value c in the expressin for the division point in such a way that it depends on the degree m of the truncated orthogonal series, which is going to be used for the approximation of the layer solution. The value c is obtained from the request that the resemblance function satisfies the differential equation in (6) at the layer point $x = 0$. This will show us how far from the layer point $x = 0$ are we alowed to go if we want to provide that the approximate solution, represented as a sum of the reduced solution and truncated orthogonal series of degree m, resembles the layer solution $u_l(x)$.

Thus, the division point should be determined using the result from the following theorem:

Theorem 1. *The value c, which determines the division point $c\varepsilon$, is*

$$c = \sqrt{\frac{-z(0)m(m-1)}{b(0,0)}}, \tag{9}$$

when ε is sufficiently small.

Proof. Introducing (8) into the differential equation in (6) we obtain

$$-\varepsilon^2 z''(x) + \frac{m(m-1)z(0)}{c^2}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-2} + b(x, r(x)) = 0.$$

At the layer point $x = 0$, with respect to $r(0) = 0$, the above equality becomes

$$-\varepsilon^2 z''(0) + \frac{m(m-1)z(0)}{c^2} + b(0,0) = 0.$$

For ε sufficiently small, this will give us the quadratic equation in c

$$\frac{m(m-1)z(0)}{c^2} + b(0,0) = 0.$$

For $z(0) > 0$ the layer solution is convex, ie. $y''(x) < 0$ for $x \in [0, c\varepsilon]$, and from (1) we can see that $\mathrm{sgn} b(x,y) = \mathrm{sgn} y''(x)$. Thus, $b(0,0) < 0$. If $z(0) < 0$ the layer solution is concave, which implies that $b(0,0) > 0$. This provides that the solution of the above equation always exists, and its positive value is given by (9).

5. Spectral approximation for the layer solution

Now we can proceed to construct the approximate solution for the problem (6) using monotone iterations combined with the spectral approximation.

We shall construct the solution of the problem (6) in the form

$$v_n(x) = u_n(x) + \frac{z(c\varepsilon)x}{c\varepsilon}, \tag{10}$$

where $u_n(x)$ is the n-th iteration of the approximate solution and we shall represent it as

$$u_n(x) = \sum_{k=0}^{m}{}' a_k T_k\left(\frac{2x}{c\varepsilon} - 1\right), \tag{11}$$

$$T_k(t) = \cos(k \cdot \arccos t), \quad k = 0, 1, \dots,$$

ie. as the truncated orthogonal series of degree m, due to the Chebysev orthogonal basis. (The notation $'a_k$ means that the summation involves $\frac{a_0}{2}$ rather than a_0.)

In order to obtain the coefficients a_k, $k = 0, 1, \dots, m$ we have to transform the layer subinterval $[0, c\varepsilon]$ into $[-1, 1]$ first, using the streching variable $t = \frac{2x}{c\varepsilon} - 1$. Thus, the finit series (11) becomes

$$w_n(t) = \sum_{k=0}^{m}{}' a_k T_k(t), \quad t \in [-1, 1] \tag{12}$$

and it represents the n-th monotone iteration for the problem

$$w''(t) + g(t, w) = 0, \quad w(-1) = 0, \; w(1) = 0, \tag{13}$$

where

$$g(t, w) = -\frac{c^2}{4} \cdot b\left(\frac{c\varepsilon}{2}(t+1), u + \frac{z(c\varepsilon)}{2}(t+1)\right).$$

If we assume that the function $g(t, w)$ is continous on $[-1, 1] \times \mathbf{R}$ and satisfies

$$K_1(v - \omega) \leq g(t, v) - g(t, \omega) \leq K_2(v, \omega), \quad v - \omega \geq 0, \quad K_1, K_2 \in \mathbf{R}, \; K_2 \leq \frac{\pi^2}{4},$$

then we can construct the the iteration sequence $w_n(t)$, defined by

$$w_n''(t) + K_1 w_n(t) = K_1 w_{n-1}(t) - g(t, w_{n-1}(t)), \tag{14}$$

$$w_n(-1) = 0, \; w_n(1) = 0,$$

starting from the arbitrary function $w_0(t)$.

If $w_0''(t) + g(t, w_0(t)) \leq 0$, $\;t \in [-1, 1]$ then $w_0(t) \geq w(t)$ and iteration sequence converges to $w(t)$ in such a way that $w_{n-1}(t) \geq w_n(t) \geq w(t)$, $n \in \mathbf{N}$. The proof can be found in [2].

In purpose to determine the coefficients of the spectral approximation in each iteration we introduce (12) into (14) and ask that the obtained equation is satisfied at Gauss-Lobatto nodes $t_i = \cos\frac{i\pi}{m}$, $i = 1, 2, \dots, m-1$. We also introduce (12) into the boundary conditions, which gives us

$$\sum_{k=0}^{m}{}' (-1)^k a_k = 0, \quad \sum_{k=0}^{m}{}' a_k = 0.$$

That procedure leads to the system of $m+1$ equations with $m+1$ unknown coefficients a_k, $k = 0, 1, \ldots, m$. The solution of this system determines the spectral approximation (11).

6. The error estimate

Out of the boundary layer, the exact solution of the problem (1),(2) is approximated by the solution of the reduced problem. The error estimate is obtained from (4) and it is given by

$$|y_\varepsilon(x) - z(x)| \le M_0 e^{\frac{-\beta x}{\varepsilon}} + M_1 e^{\frac{-\beta(1-x)}{\varepsilon}} + M_2\varepsilon^2. \tag{15}$$

Let us now estimate the error upon the layer subinterval $(0, c\varepsilon]$. The error function, according to (5) and (1)-(13) is

$$d(x) = |y_\varepsilon(x) - v_n(x)| \le |y_\varepsilon(x) - u_l(x)| + |w(t) - w_n(t)|. \tag{16}$$

In order to estimate the first term we have to prove the following lemma:

Lemma 2. *Let* $b(x,y) \in \mathbf{C}^2([0,c\varepsilon] \times \mathbf{R})$ *and* $b_y(x,y) \ge \beta^2$, $\beta \in \mathbf{R}$ *for* $x \in (0, c\varepsilon]$. *Then*

$$|y_\varepsilon(x) - u_l(x)| \le M(\varepsilon^2 + e^{-\beta c}) \quad \textit{for} \quad x \in (0, c\varepsilon], \tag{17}$$

where M *is arbitrary constant independent of* x *and* ε.

Proof. The function $u_\varepsilon(x)$ satisfies the boundary value problem

$$L_\varepsilon y_\varepsilon(x) = 0,\ x \in (0, c\varepsilon],\ y_\varepsilon(0) = 0,\ y_\varepsilon(c\varepsilon) = y_0. \tag{18}$$

Subtracting (6) from (18) we obtain

$$L_\varepsilon(y_\varepsilon - u_l)(x) = 0,\ (y_\varepsilon - u_l)(0) = 0,\ (y_\varepsilon - u_l)(c\varepsilon) = y_\varepsilon(c\varepsilon) - z(c\varepsilon).$$

Under the given assumptions the operator L_ε is inverse monotone. So, by the principle of inverse monotonicity we can conclude that

$$|y_\varepsilon(x) - u_l(x)| \le |y_\varepsilon(c\varepsilon) - z(c\varepsilon)|. \tag{19}$$

Using the estimate (15) for $x = c\varepsilon$, as $e^{\frac{-\beta(1-c\varepsilon)}{\varepsilon}}$ tends to zero when ε is sufficiently small, we obtain (17).

In order to estimate the second term we use the fact that, starting from $w_0(t) \equiv 0$, because of $g(t, w_0(t)) \leq 0$, each iteration $w_n(t)$ represents the upper bound for the solution of the problem (13). The lower solution $\omega_n(t)$, as shown in [2], represents the solution of the problem

$$\omega_n''(t) + K_2\omega_n(t) = K_2 w_n(t) - g(t, w_n(t)), \;\; \omega_n(-1) = \omega_n(1) = 0,$$

which can be determined in the form of the appropriate truncated Chebyshev series using the standard procedure.

Using these results, we come to the error estimate, which is given by the following theorem:

Theorem 2. *Let $\omega_n(t)$ represent the lower solution and $w_n(t)$ is the n-th monoton iteration. The error $d(x)$, defined by (16), can be estimated as*

$$|d(x)| \leq M(\varepsilon^2 + e^{-\beta c}) + \left| w_n\left(\frac{2x}{c\varepsilon} - 1\right) - \omega_n\left(\frac{2x}{c\varepsilon} - 1\right)\right|, \tag{20}$$

Proof. Using (16),(19) and the inequality $\omega_n(t) \leq w(t) \leq w_n(t)$ we directly obtain (20).

7. Numerical example

We shall use the following test example:

$$-\varepsilon^2 y'' + y^2 - f(x) = 0\;,\; y(0) = y(1) = 0,$$

with the exact solution

$$y_\varepsilon(x) = \frac{(2x+1)(e^{\frac{1-x}{\varepsilon}} - e^{\frac{x}{\varepsilon}})}{e^{\frac{1}{\varepsilon}} - 1} + 1.$$

The difference between the exact solution and the approximate one inside the boundary layer is given in the following table:

	m=4	m=6	m=8
c	3.46	5.48	7.48
n=1	0.1	0.7	1.5
n=2	0.015	0.04	0.06
n=3	0.004	0.008	0.014
n=4	—	0.0017	0.0027
n=5	—	—	0.0006

The results are given for $\varepsilon = 2^{-16}$, using truncated Chebyshev series of degree $m = 4$, $m = 6$ and $m = 8$. The calculation is performed in five iterations.

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