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A NOTE ON BOOLEAN LATTICES OF FINITE POSETS

Branimir Šešelja, Andreja Tepavčević
Institute of Mathematics,
University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad,
Yugoslavia

Abstract

It is proved that the collection of all finite partially ordered sets with the same poset of meet-irreducible elements is a finite Boolean algebra, and that every finite Boolean algebra can be represented by such collection. In addition, we give necessary and sufficient conditions under which a lattice of all lattices determined by the same poset of meet-irreducibles is a sublattice of the mentioned Boolean lattice.

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1. Preliminaries

The present paper is based on the well known result of G. Birkhoff [2] by which there is a one-to-one correspondence between all finite partially ordered sets and all finite distributive lattices. Namely, every finite partially ordered set X is isomorphic to the poset of meet-irreducible elements of a finite distributive lattice. As it is known, this lattice is, up to an isomorphism, the lattice of all order filters of X, endowed with the dual of inclusion as partial order; the subset of principal filters is order-isomorphic with X.

Obviously, X can be isomorphic to the posets of meet-irreducibles of some other, non-distributive lattices, and also to the posets of meet-irreducibles of some finite posets. In the paper [10] it was proved that all these posets and lattices can be considered as particular subsets of the corresponding distributive lattice. In fact, in this paper necessary and sufficient conditions under which a finite partially ordered set and a finite distributive lattice have isomorphic posets of meet-irreducibles were given.

In the paper [9], the collection $\mathcal{L}(X)$ of all finite lattices generated in the above sense by the poset X of meet-irreducibles was investigated. It was proved that this collection is a lattice. Necessary and sufficient conditions under which this lattice is modular, distributive and Boolean were also given.

Representations of some finite lattices by the collection-lattice $\mathcal{L}(X)$ were given in the paper [9]. A part of these investigations was motivated by some functional representations of finite lattices, given in [7].

In the present paper, the collection $\mathcal{C}(X)$ of all finite posets (including lattices) generated by the same poset X of meet-irreducibles is investigated. We prove that this collection is a finite Boolean lattice. Such a representation exists for every finite Boolean lattice. We also give necessary and sufficient conditions under which $\mathcal{L}(X)$ is a sublattice of $\mathcal{C}(X)$.

2. Results

An element x of a partially ordered set $(P, \leq)$ is said to be **meet-irreducible**, if it is different from the greatest element, the top (provided that it exists) and if it satisfies the following condition:

for any $y, z \in P$, if $x = inf\{y, z\}$ then $x = y$ or $x = z$

(obviously, x is meet-irreducible also in the case when there are no elements for which it is the infimum).

Recall that a meet-irreducible element x of a lattice L is the one which is different from the top element and satisfies the implication:

if $x = y \wedge z$ then $x = y$ or $x = z$.

Throughout the paper, $(P, \leq)$ is supposed to be a finite partially ordered set (finite poset), whose top element, if it exists, is denoted by 1 (or 1_P, in the case when a lattice L is also considered, whose top element is then denoted by 1_L).

The following lemma is a result from the paper [10]. Actually, it proves that every finite partially ordered set with a given poset X of meet-irreducibles can be considered as a particular subset of a distributive lattice whose poset of meet-irreducibles is order isomorphic with X.

Let L be a finite distributive lattice for which $(Y, \leq)$ is the poset of meet-irreducible elements. Further on, let Y' be a subset of L constructed as follows. If $y \in Y$ is such that $y = inf_Y Z$, where $Z = \{z \in Y \mid y < z\}$. (i.e., y is the infimum in Y of the set of all meet-irreducible elements above it), then, we define $y' := \bigwedge_L Z$ (hence, y' is the meet of the set of all meet-irreducibles above y, this time in L); if y is the only co-atom in L (i.e., if $L \setminus \{1\}$ is a lattice with the top element y), then let $y' := 1$. Finally, let

$$Y' = Y \cup \{a \in L \mid a = y' \text{ for some } y \in Y\}. \qquad (1)$$

Lemma 1. [10] *A finite partially ordered set $(P, \leq)$ and a finite distributive lattice L have isomorphic posets of meet-irreducible elements (X and Y, respectively) if and only if $(P, \leq)$ is isomorphic to a sub-poset of L which contains Y' (defined by* (1)*)* □.

Hence, every finite poset $(Y, \leq)$ generates a collection of partially ordered sets (some of them are lattices) for which $(Y, \leq)$ is a poset of meet-irreducible elements. Namely, each of these partially ordered sets (lattices) is a subset of the distributive lattice whose poset of meet-irreducibles is $(Y, \leq)$. Some of these posets can be isomorphic, under the mapping which extends an automorphism of the poset Y. On the other hand, (as proved in [10]), every partially ordered set whose poset of meet-irreducibles is isomorphic to $(Y, \leq)$ is embeddable into the corresponding distributive lattice. In the following, we investigate the above mentioned collection of posets generated by Y.

Let $(Y, \leq)$ be a finite partially ordered set, L the distributive lattice in which $(Y, \leq)$ is a poset of meet-irreducible elements, and $\mathcal{C}(Y)$ a collection of subsets of L, containing Y' (defined by (1)) and which are posets under the order inherited from L i.e.,

$$\mathcal{C}(Y) := \{P \subseteq L \mid Y' \subseteq P, P \text{ is a poset under the order from } L\}.$$

It is obvious that $\mathcal{C}(Y)$ is a partially ordered set under the set inclusion, which we denote by $(\mathcal{C}(Y), \subseteq)$.

Theorem 1. *For any finite poset* $(Y, \leq)$*, the partially ordered set* $(\mathcal{C}(Y), \subseteq)$ *is a Boolean lattice.*

Proof. We prove that $(\mathcal{C}(Y), \subseteq)$ is a lattice. There is a top element in the collection, the distributive lattice L. Further on, $\mathcal{C}(Y)$ is closed under the set intersection. Indeed, if P_1 and P_2 are two posets from $\mathcal{C}(Y)$, then Y' is a subset of both of them, hence Y' is contained in their intersection. By Lemma 1, $P_1 \cap P_2$ also belongs to $\mathcal{C}(Y)$. Thus, $(\mathcal{C}(Y), \subseteq)$ is a lattice.

This lattice is Boolean, since by Lemma 1, For every subset Z of $L \setminus Y'$, $Y' \cup Z$ belongs to $\mathcal{C}(Y)$. Indeed, by Lemma 1 the only condition which a poset has to satisfy in order to belong to the collection is to be a subset of L containing Y'. Whence, the lattice $(\mathcal{C}(Y), \subseteq)$ is isomorphic with the power set of $L \setminus Y'$. □

It is possible to represent every finite Boolean lattice by the suitable collection-lattice, as proved in the sequel.

Theorem 2. *Every finite Boolean lattice with more than 2 elements is isomorphic to the collection lattice* $(\mathcal{C}(Y), \subseteq)$ *of some finite poset* $(Y, \leq)$.

Proof. We give one possible construction of the poset Y, such that $(\mathcal{C}(Y), \subseteq)$ is isomorphic to a given finite Boolean lattice.

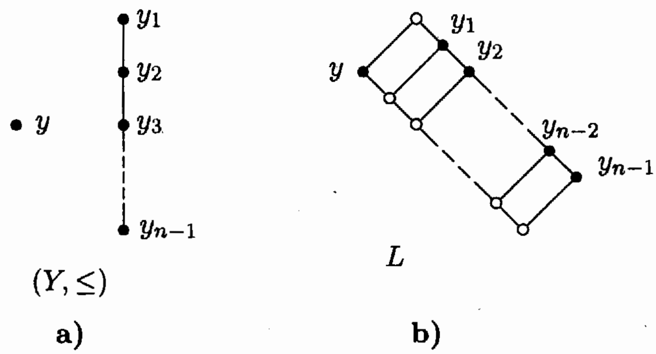

Fig. 1

For $n = 2, 3, 4, \ldots$, let $(Y, \leq)$ be a disjoint union of two chains, with 1 and $n-1$ elements, respectively (Figure 1.a)). The corresponding distributive lattice in which Y is the poset of meet-irreducibles is represented in Figure 1.b). Being a disjoint union of two chains, the set Y does not contain infima of non-comparable elements, hence by (1), $Y' = Y$. Further, $|L \setminus Y'| = n$. Hence, the lattice $(\mathcal{C}(Y), \subseteq)$ which is Boolean by Theorem 1, has exactly 2^n elements. □

In the papers [8,9] the foregoing problems were discussed for lattices. Namely, it is proved there that the collection $\mathcal{L}(Y)$ of all finite lattices determined by the same poset $(Y, \subseteq)$ of meet-irreducibles is a lattice. As in the present paper, these lattices are taken to be subsets of a distributive one, whose poset of meet-irreducibles is also $(Y, \leq)$. Hence, the collection $\mathcal{L}(Y)$ of lattices whose poset of meet-irreducibles is $(Y, \leq)$ is a subset of the collection $\mathcal{C}(Y)$ of all partially ordered sets with the same property: in each of them $(Y, \leq)$ is a poset of meet-irreducibles. As proved in the present article, $(\mathcal{C}(Y), \subseteq)$ is always a Boolean lattice, which is not the case with the lattice $(\mathcal{L}(Y), \subseteq)$; this one can be Boolean, but also only distributive, and even a non-modular lattice. In the following theorem we give conditions under which $(\mathcal{L}(Y), \subseteq)$ is a sublattice of the Boolean lattice $(\mathcal{C}(Y), \subseteq)$.

Recall that an element of a lattice L is **reducible** if it is not meet, nor join-irreducible. If Y is a poset of meet-irreducible elements in a finite distributive lattice L and $a \in L$, then (as defined in [9]) we shall say that a is Y-**independent** if it is not in Y, nor it is a meet of elements from Y which are greater than an $y \in Y$, nor it is a join of meet-irreducible elements.

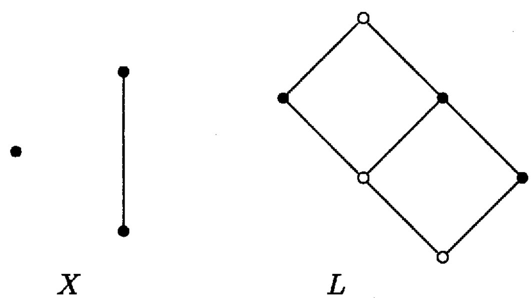

Fig. 2

The following lemma is a result from the paper [9].

Lemma 2. [9] *$(\mathcal{L}(Y), \subseteq)$ is a distributive lattice if and only if the distributive lattice L from the collection does not contain an Y-independent reducible element, which is a join of other Y-independent elements.* □

Remark. The "if part" of the proof of the above proposition is based on the fact that the join in $(\mathcal{L}(Y), \subseteq)$, under the assumed conditions, coincides with the set-union.

Theorem 3. *Let $(Y, \leq)$ be a finite poset and L the distributive lattice whose poset of meet-irreducibles is $(Y, \leq)$. Then the lattice $(\mathcal{L}(Y), \subseteq)$ is a sublattice of the lattice $(\mathcal{C}(Y), \subseteq)$ if and only if L does not contain an Y-independent reducible element which is a join of other Y-independent elements.*

Proof.

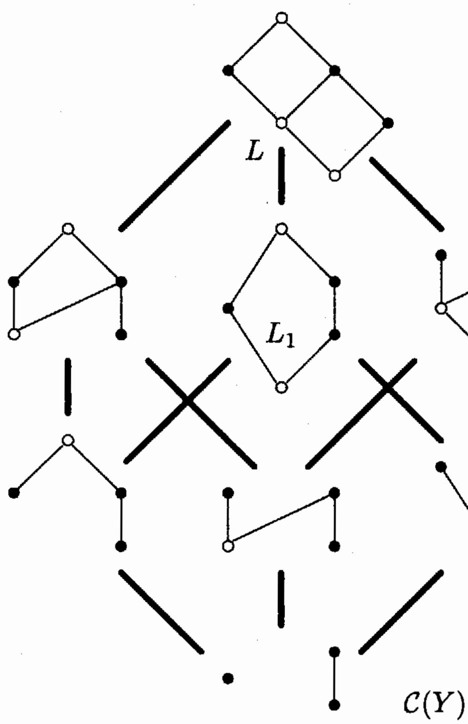

Fig. 3

If $(\mathcal{L}(Y), \subseteq)$ is a sublattice of $(\mathcal{C}(Y), \subseteq)$, then obviously the former is distributive and by Lemma 2 the foregoing condition is satisfied. On the other hand, if this condition is satisfied, then by the above remark, the join in $(\mathcal{L}(Y), \subseteq)$ is defined by the set-union, which is also the case in $(\mathcal{C}(Y), \subseteq)$. Since the meet is the set-intersection in both lattices, it follows that $(\mathcal{L}(Y), \subseteq)$ is a sublattice of $(\mathcal{C}(Y), \subseteq)$. □

Example. The finite poset Y is the poset of meet-irreducibles of the distributive lattice L (Fig. 2). All partially ordered sets for which Y is the poset of meet-irreducibles are considered to be subsets of L. The collection $(\mathcal{C}(Y), \subseteq)$ of these posets is a Boolean lattice (Fig. 3).

Note that there is no non-trivial automorphism on the poset X. Hence, $\mathcal{C}(Y)$ consists of all, up to the automorphism different partially ordered sets for which X is a poset of meet-irreducibles. Further, there are two lattices in the collection. In Figure 3 they are denoted by L and L_1. Since L satisfies conditions of Theorem 3, $(\mathcal{L}(Y), \subseteq)$ is a (two-element) distributive sublattice of $(\mathcal{C}(Y), \subseteq)$.

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