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ON A COMPLETION OF POSETS BY FUZZY SETS

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Abstract

Using representations of posets and lattices by levels of fuzzy sets, we obtain a completion of posets to complete lattices. It turns out that this completion is equivalent with the famous Dedekind-MacNeille completion, but the algorithm presented here is much simpler.

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1. Preliminaries

Most of the notations and notions are taken from [1]. Let $(P, \leq)$ be a partially ordered set (poset), denoted also by the underlying set P only.

The **bottom** element of a poset, if it exists, is denoted by 0; similarly, the **top** element is denoted by 1. The greatest lower bound (infimum) of x and y, if it exists, is denoted by $x \wedge y$, and is said to be their **meet**. Similarly, supremum of x and y, if it exists, is their **join**, $x \vee y$. If $S \subseteq P$, the set of all upper bounds of S is denoted by S^u, and the set of all lower bounds is denoted by S^ℓ:

$$S^u := \{x \in P \mid (\forall s \in S) s \leq x\}$$

and

$$S^\ell := \{x \in P \mid (\forall s \in S) s \geq x\}.$$

If $x \in P$, then the ideal generated by x is denoted by $\downarrow x$:

$$\downarrow x := \{y \in P \mid y \leq x\}.$$

Dually, the filter generated by x is denoted by $\uparrow x$:

$$\uparrow x := \{y \in P \mid y \geq x\}.$$

If supremum and infimum exist for each pair x, y of elements in a poset, then this poset is a lattice and it is denoted by L. Obviously, join ($\vee$) and meet ($\wedge$) are binary operations in every lattice. Recall that lattice L is **complete** if for every $S \subseteq L$, there exist supremum and infimum.

If P is a poset, L a complete lattice and $\varphi : P \longrightarrow L$ an order embedding (i.e., an injection preserving the order), than L is a **completion** of P.

If $(P, \leq)$ is a poset, then

$$DM(P) := \{Q \subseteq P \mid Q^{u\ell} = Q\}$$

is a complete lattice under inclusion, and it is **Dedekind-MacNeille completion** of P. In this case

$$\varphi(x) = \downarrow x.$$

$DM(P)$ preserves all suprema and infima existing in P, i.e., if $Q \subseteq P$ and $\bigvee Q$ exists in P, then $\varphi(\bigvee Q) = \bigvee \varphi(Q)$, and analogously for infimum (note that $\varphi(Q) = \{\varphi(q) \mid q \in Q\}$).

Fuzzy sets

If $(P, \leq)$ is a partially ordered and S a nonempty set, then a function $\overline{A} : S \to P$ is a **partially ordered valued fuzzy set** (**poset valued fuzzy set**, P-**fuzzy set**) on S. For every $p \in P$, the following subset of S is called a p-**cut** or a **level subset** (**level**) of $\overline{A}$: $A_p := \{x \in S \mid \overline{A}(x) \geq p\}$.

The set of all levels of a fuzzy set $\overline{A} : S \longrightarrow P$ is denoted by A_P. Thus,

$$A_P := \{A_p \mid p \in P\}.$$

A_P is a poset under inclusion.

If L is a poset which is a lattice, a fuzzy set $\overline{A} : S \longrightarrow L$ is said to be a **lattice valued fuzzy set** (**L-fuzzy set**).

Some of the properties of P and L-fuzzy sets are listed in the sequel. It is obvious that properties of P-fuzzy sets are valid for L-fuzzy sets, as well.

I *Let $\overline{A} : S \to P$ be a P-fuzzy set on S. Now,*

a) if $p, q \in P$ and $p \le q$, then $A_q \subseteq A_p$.

b) for any $x, y \in S$, $\overline{A}(x) \le \overline{A}(y)$ if and only if $A_{\overline{A}(y)} \subseteq A_{\overline{A}(x)}$.

II *For every $x \in S$,*

$$\bigcap(A_p \in A_P \mid x \in A_p) \in A_P; \tag{1}$$

and

$$\bigcup(A_p \in A_P) = S.$$

The family of sets satisfying condition (1) is said to be **closed under componentwise intersections**.

III *If $\mathcal{F}$ is a family of subsets of S closed under componentwise intersections, such that the union of $\mathcal{F}$ is S, then there is a P-fuzzy set $\overline{A} : S \longrightarrow \mathcal{F}$, defined by:*

$$\overline{A}(x) = \bigcap(Y \in \mathcal{F} \mid x \in Y),$$

such that

$$A_p = p,$$

for all $p \in \mathcal{F}$.

IV *Let $\overline{A} : S \longrightarrow L$ be an L-fuzzy set on S. A family of level sets A_L is closed under intersections, and contains S.*

V *If $\mathcal{F}$ is a family of subsets of S closed under intersections which contains S, then there is an L-fuzzy set $\overline{A} : S \longrightarrow \mathcal{F}$, defined by:*

$$\overline{A}(x) = \bigcap(Y \in \mathcal{F} \mid x \in Y),$$

such that

$$A_p = p,$$

for all $p \in \mathcal{F}$.

2. Completion

Every poset can be represented by an isomorphic collection of levels of the particular fuzzy set, as follows.

Theorem 1. *Let P be a partially ordered set. Let $\overline{A} : P \longrightarrow P$ be defined by $\overline{A}(x) = x$, for all $x \in P$. Then, P is anti-isomorphic with A_P, under $p \mapsto A_p$.*

(Anti-isomorphism is isomorphism of P with A_P ordered dually to inclusion.)

Proof. By the construction, every element of P belongs to the collection of images of the function $\overline{A}$. Hence, by I b), P and A_P are order isomorphic posets. □

Since two lattices are isomorphic if they are isomorphic as posets, this theorem is valid also for lattices.

A completion of an arbitrary poset (possible a lattice which is not complete), to a complete lattice by fuzzy sets, runs as follows.

Algorithm

Let P be a poset. We make a representation by level sets, defining a fuzzy set $\overline{A} : P \to P$ as in Theorem 1. By (II) the collection of level sets of this fuzzy set is a family of subsets of the domain P of $\overline{A}$, closed under componentwise intersections. Now, we complete this collection of subsets by all missing set intersections, and add the domain P to the collection. In such a way we obtain a lattice of subsets and a corresponding fuzzy set as in (V). This lattice is a lattice of levels of the obtained fuzzy set, and a completion of poset P. □

The above algorithm by which every poset can be completed to a complete lattice is a **completion by fuzzy sets.**

Example 1.

Let P be a fuzzy set given by its Hasse diagram in Fig. 1. $P = \{a, b, c, d\}$, and a fuzzy set representing P is $\overline{A} : P \longrightarrow P$,

$$\overline{A} = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}.$$

The corresponding level sets are: $\overline{A}_a = \{a\}; \overline{A}_b = \{b\}; \overline{A}_c = \{a, b, c\}; \overline{A}_d = \{a, b, d\}$. Family $\overline{A}_P$ is closed under componentwise intersections. In order to obtain a completion of this family, we add all intersections of elements from $\overline{A}_P$: $\{a, b\}$, $\emptyset$ and set P. In this way we obtain the collection $\{\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$ which is a lattice under inclusion.

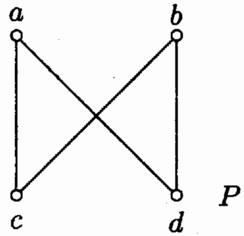

Figure 1

The obtained completion is the lattice L dual to the above collection lattice and it is represented in Fig 2.

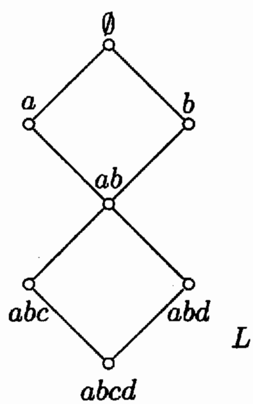

Figure 2

□

In the following we prove that the completion of posets by fuzzy sets preserves all existing suprema and infima. More precisely, we show that though the procedure is completely different, the obtained lattice is the same as Dedekind MacNeille completion.

Theorem 2. *Let P be a partially ordered set, and L_P the lattice obtained by the completion of P by fuzzy sets (i.e., by the above algorithm). This completion is same as Dedekind-MacNeille completion.*

Proof. Let P be a partially ordered set. Let $\overline{A} : A \longrightarrow P$ be a fuzzy set such that its family $\mathcal{F}$ of level sets, is under inclusion the poset anti-isomorphic with P (such fuzzy set exists, by Theorem 1.) Family $\mathcal{F}$ is closed under componentwise intersections, and its set union is the domain of $\overline{A}$. Let L be a family of all intersections of all subfamilies of $\mathcal{F}$ plus A:

$$L = \{\bigcap \mathcal{F}_1 \mid \mathcal{F}_1 \subseteq \mathcal{F}\} \cup \{A\}.$$

Since L is a family of subsets of set A closed under intersections containing set A, L is a lattice under inclusion. Let $\mathcal{L} = (L, \leq)$ be a lattice anti-isomorphic with $(L, \subseteq)$. This lattice is a completion of P, since $\mathcal{F}$ is anti-isomorphic with P.

If the dual of L is a completion of the dual of P, than L is a completion of P, as well.

We have that for a $p \in P$, level A_p is equal to $\uparrow p$, i.e., in the dual of P, $A_p = \downarrow p$.

Now we prove that $L = DM(P)$. Recall that

$$DM(P) = \{X \subseteq P \mid X^{u\ell} = X\}, \text{ and}$$

$$L = \{\bigcap \{\downarrow a \mid a \in X \subseteq P\}\} \cup \{P\}.$$

Since $\{\downarrow x\}^{u\ell} = \downarrow x$, we have that $\downarrow x \in DM(P)$.

To prove that $L \subseteq DM(P)$, let $X \subseteq P$. An element of L is P or it is of the form

$$\bigcap \{\downarrow a \mid a \in X\} = \{x \in P \mid (\forall a \in X) x \leq a\}$$

$= X^\ell$.

Since $P^{u\ell} = P$ and $X^\ell = X^{\ell u \ell}$, it follows that in both cases we obtain an element from $DM(P)$, hence $L \subseteq DM(P)$.

On the other hand, if $X \subseteq P$, such that $X \in DM(P)$, i.e., $X^{u\ell} = X$, then

$$X^{u\ell} = \bigcap \{\downarrow a \mid a \in X^u\} \in L.$$

Hence, $DM(P) \subseteq L$, completing the proof. □

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