Novi Sad J. Math.
Vol. 29, No. 2 1998,79-90

CONSTRUCTING MAXIMAL BLOCK-CODES BY BISEMILATTICE VALUED FUZZY SETS

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Abstract

A bisemilattice valued fuzzy set generates two collections of level sets, corresponding to each of two orders existing in a bisemilattice. Set union of these two collections is considered to be a binary block-code. Starting with a finite bisemilattice, we present an algorithm for the construction of a bisemilattice valued fuzzy set which has the following properties: it has maximal number of levels (i.e., maximal cardinality of the corresponding block-code) and minimal domain.

AMS Mathematics Subject Classification (1991): 04A72, 06A12
Key words and phrases: fuzzy sets, bisemilattices

Bisemilattices

A **bisemilattice** $\mathcal{B} = (B, \wedge, \vee)$ is an algebra of type $(2,2)$, where $(B, \wedge)$ and $(B, \vee)$ are commutative and idempotent semigroups, i.e., semilattices.

There are two orderings corresponding to a bisemilattice $(B, \wedge, \vee)$:

$x \leq_\vee y$ if and only if $x \vee y = y$ and

$x \leq_\wedge y$ if and only if $x \wedge y = x$.

Further, $x \geq_\vee y$ stands for $y \leq_\vee x$, and $x \geq_\wedge y$ for $y \leq_\wedge x$.

Starting with two orderings, a bisemilattice $(B, \wedge, \vee)$ can also be considered as a relational system, as follows. $(B, \leq_\vee, \leq_\wedge)$ is a **bisemilattice**, if B is a nonempty set and $\leq_\vee$, $\leq_\wedge$ are ordering relations on B, such that $(B, \leq_\vee)$ is a join-semilattice ($\vee$-semilattice), i.e., the poset in which every two-element subset has the least upper bound, and $(B, \leq_\wedge)$ is a meet-semilattice ($\wedge$-semilattice) in which each two-element subset has the greatest lower bound.

A diagram of a bisemilattice consists of two Hasse diagrams, one for each ordering. In drawing diagrams, the following convention is used: if $x \leq_\vee y$ and $z \leq_\wedge t$, then x is below y, and z below t.

Meet-irreducible elements

An element a of a poset $(P, \leq)$ is said to be **meet-irreducible** if it is different from the top element (if it exists in P) and if the following holds:

$a = inf\{b, c\}$ implies $a = b$ or $a = c$.

The notion of a **join-irreducible** element is introduced dually.

An element of a semilattice is meet-irreducible (join-irreducible) if it is meet-irreducible (join-irreducible) in the semilattice considered as a poset (under the induced ordering).

I *Every element of a finite meet-semilattice (join-semilattice) is equal to a meet (join) of some meet-irreducible (join-irreducible) elements.*

We say that an element a of a bisemilattice $(B, \vee, \wedge)$ is **meet-irreducible** if it is meet-irreducible in *at least one* of semilattices $(B, \vee)$ and $(B, \wedge)$.

We use the notion of a **principal filter** $a\uparrow$ in a poset $(P, \leq)$: for $a \in P$,

$a\uparrow := \{x \in P \mid a \le x\}$.

Throughout the paper, if $(B, \vee, \wedge)$ is a bisemilattice, then principal filters on semilattices $(B, \vee)$ and $(B, \wedge)$ are considered as sets (not as posets).

Fuzzy sets

The notion of a fuzzy set is well known. We advance some notations and properties (see [3]). Let P be an ordered structure (real interval, lattice, poset, etc.) with the order $\le$ and $A \neq \emptyset$ a set. Then, the map $\overline{A} : A \to P$ is a P-fuzzy subset of A (or: a P-fuzzy set on A). For $p \in P$, a **level set** (**level**, **p-cut**) of $\overline{A}$ is a (crisp) subset A_p of A, such that $x \in A_p$ if and only if $\overline{A}(x) \ge p$. The characteristic function of A_p is denoted by $\overline{A}_p$ and is called the **level function** of $\overline{A}$. As usual,

$\overline{A}(X) := \{p \in P \mid p = \overline{A}(x) \text{ for some } x \in X\}$.

If $\overline{A}$ is a P-fuzzy set on X, then binary relation $\approx$ on P, defined by

$p \approx q$ if and only if $A_p = A_q$,

is an equivalence relation on P.

II ([7]) *If $\overline{A}$ is a P-fuzzy set on X, then:*

a) for any $p, q \in P$, $p \approx q$ if and only if $p\uparrow \cap \overline{A}(x) = q\uparrow \cap \overline{A}(x)$;

b) if $[p]_\approx$ is the $\approx$-class to which p belongs, then the relation $\le$, defined by

$[p]_\approx \le [q]_\approx$ *if and only if* $A_q \subseteq A_p$

is an order on $P/\approx$;

c) if $p = \overline{A}(x)$ for some $x \in X$, then $p = \bigvee [p]_\approx$.

The following two propositions are proved in [6].

III *Let $\overline{A} : X \to P$ be a fuzzy set on X, where P is a finite meet-semilattice. All the p-cuts of $\overline{A}$ are distinct if and only if the following hold: at most one meet-irreducible element of P is not in $\overline{A}(X)$, and such an element, if it exists, is maximal in P; further on, all meet irreducible elements from the semilattice $P \setminus \{m\}$ also belong to $\overline{A}(X)$.*

IV *Let $\overline{A} : X \to P$ be a fuzzy set on X, where P is a finite join-semilattice. All the p-cuts of $\overline{A}$ are distinct if and only if all meet irreducible elements from the semilattice P different from the greatest element belong to $\overline{A}(X)$.*

Bisemilattice-Valued Fuzzy Sets

A **bisemilattice valued fuzzy set** (B-fuzzy set) is introduced in ([3]) as a mapping $\overline{A} : X \longrightarrow B$ from a nonempty set X to a bisemilattice $\mathcal{B} = (B, \wedge, \vee)$.

For each $p \in B$, there are two level subsets defined as follows:

$A_p^{\vee} = \{x \in X \mid \overline{A}(x) \geq_{\vee} p\}$ and

$A_p^{\wedge} = \{x \in X \mid \overline{A}(x) \geq_{\wedge} p\}$.

The corresponding level functions are:

$\overline{A}_p^{\vee}(x) = 1$ if and only if $\overline{A}(x) \geq_{\vee} p$ and

$\overline{A}_p^{\wedge}(x) = 1$ if and only if $\overline{A}(x) \geq_{\wedge} p$.

Thus, for a B-fuzzy set $\overline{A} : X \longrightarrow B$, there are two families of level subsets:

$A_B^{\vee} = \{A_p^{\vee} \mid p \in B\}$, and $A_B^{\wedge} = \{A_p^{\wedge} \mid p \in B\}$.

Obviously, a bisemilattice valued fuzzy set determines two semilattice valued fuzzy sets with the same domain and co-domain. Hence, we have the following properties.

If $\overline{A} : X \to B$ is a bisemilattice valued fuzzy set, then relations $\approx_{\vee}$ and $\approx_{\wedge}$, defined on B by

$p \approx_{\vee} q$ if and only if $A_p^{\vee} = A_q^{\vee}$,

$p \approx_{\wedge} q$ if and only if $A_p^{\wedge} = A_q^{\wedge}$,

are equivalence relations.

For $p \in B$ let $[p]_{\approx_{\vee}}$ and $[p]_{\approx_{\wedge}}$ be the corresponding equivalence classes. Then, $p \to \bigvee^{\vee}[p]_{\approx_{\vee}}$ and $p \to \bigvee^{\wedge}[p]_{\approx_{\wedge}}$ are closure operations on bisemilattices $(B, \leq_{\vee})$ and $(B, \leq_{\wedge})$ respectively.

The following proposition is an immediate consequence of the fact that a bisemilattice valued fuzzy set determines two semilattice valued ones. Therefore, the proof follows by the corresponding proposition in [7].

Proposition 1. *Let $\overline{A}$ be a B-fuzzy set on X. Then:*

a) if there is the smallest element (0) in $(B, \wedge)$, then $A_0^{\wedge} = X$;

b) if $p \leq_{\vee} q$, then $A_q^{\vee} \subseteq A_p^{\vee}$, and if $p \leq_{\wedge} q$, then $A_q^{\wedge} \subseteq A_p^{\wedge}$;

c) for every $x \in X$,

$\overline{A}(x) = \bigvee^{\vee}\{p \in B \mid \overline{A}_p^{\vee} = 1\}$ *and* $\overline{A}(x) = \bigvee^{\wedge}\{p \in B \mid \overline{A}_p^{\wedge} = 1\}$ *(i.e., in both equalities the join on the right exists and is equal to $\overline{A}(x)$).* □

Results: *B*-fuzzy Sets and Codes

Let B be a finite bisemilattice and $S = \{1, \ldots, n\}$. To every B-fuzzy set on S there corresponds a binary block-code V of length n, determined in the following way.

For $p \in B$, let $v_p^{\vee} := x_1 \ldots x_n$, so that $x_i := \overline{A}_p^{\vee}(i)$, $i = 1, \ldots, n$. Let $V^{\vee} := \{v_p^{\vee} \mid p \in B\}$. Obviously, each $v_p^{\vee}$ can be identified with $\overline{A}_p^{\vee}$, since both represent the characteristic function of the p-cut, level set $A_p^{\vee}$. Therefore there is a bijective correspondence between $\approx_{\vee}$-classes $[p]_{\approx_{\vee}}$ of $\overline{A}$ and code-words $v_p^{\vee}$. Consequently, relation $\leq_{\vee}$ on $V^{\vee}$, defined by

$v_p^{\vee} \leq_{\vee} v_q^{\vee}$ if and only if $v_q^{\vee}(i) \leq v_p^{\vee}(i)$ for every $i \in \{1, \ldots, n\}$,

is a partial order (for example, $11101 \leq_{\vee} 00001$).

Replacing $\vee$ by $\wedge$ and repeating the same procedure in an analogous way, we obtain the set $V^{\wedge} := \{v_p^{\wedge} \mid p \in B\}$, where $v_p^{\wedge} = x_1 \ldots x_n$ corresponds to $\overline{A}_p^{\wedge}$.

Let $V_{\overline{A}} := V^{\vee} \cup V^{\wedge}$. V is the block-code which **corresponds** to $\overline{A}$. We recall that by the construction *the number of code-words in $V_{\overline{A}}$ coincides with the number of levels of $\overline{A}$*:

(1) $$|V_{\overline{A}}| = |A_B^{\vee} \cup A_B^{\wedge}|.$$

Example 1.

Semilattice $(B, \vee, \wedge)$ is given by two diagrams in Figure 1.

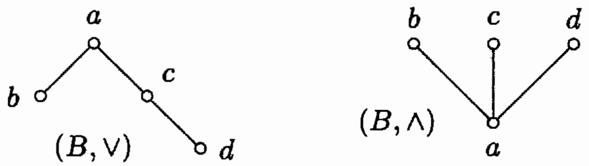

Fig. 1

Let

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 \\ b & c & d \end{pmatrix}$$

be a B-fuzzy set on $S = \{1, 2, 3\}$. Then

$A_B^{\vee} = \{\emptyset, \{1\}, \{2\}, \{2,3\}\}$ and $A_B^{\wedge} = \{\{1\}, \{2\}, \{3\}, \{1,2,3\}\}$.

Hence,

$V^{\vee} = \{000, 100, 010, 011\}$ and $V^{\wedge} = \{100, 010, 001, 111\}$, and thus

$V_{\overline{A}} = \{000, 001, 010, 100, 011, 111\}$. □

In the following we discuss the problem of finding codes of maximal cardinality induced by B-fuzzy sets for a given bisemilattice B. We also present an algorithm for the construction of such fuzzy sets and of the corresponding codes.

Let $(B, \vee, \wedge)$ be a bisemilattice. If $p \in B$, then $p{\uparrow^{\vee}}$ and $p{\uparrow^{\wedge}}$ denote principal filters generated by p in $(B, \vee)$ and $(B, \wedge)$, respectively. Recall that these filters are here considered as sets, without any ordering. Further, let

$$\mathcal{F}^{\vee} := \{p{\uparrow^{\vee}} \mid p \in B\} \quad \text{and} \quad \mathcal{F}^{\wedge} := \{p{\uparrow^{\wedge}} \mid p \in B\}.$$

In other words, $\mathcal{F}^{\vee}$ and $\mathcal{F}^{\wedge}$ are collections of principal filters on $(B, \vee)$ and $(B, \wedge)$, respectively.

Lemma 1. $|\mathcal{F}^{\vee}| = |\mathcal{F}^{\wedge}| = B$.

Proof. Obvious, since every $p \in B$ generates a filter in each semilattice. □

Recall that for $\overline{A} : S \to B$,

$\overline{A}(S) = \{p \in B \mid p = \overline{A}(x) \text{ for some } x \in S\}$.

Lemma 2. *Let $\overline{A} : S \to B$ be a B-fuzzy set on S, and $p, q \in B$. If $p{\uparrow^{\vee}}$*

$\cap \overline{A}(S) = q\uparrow^{\wedge} \cap \overline{A}(S)$, *then the corresponding code-words* $v_p^{\vee}$ *and* $v_q^{\wedge}$ *are equal.*

Proof. $p\uparrow^{\vee} \cap \overline{A}(S) = q\uparrow^{\wedge} \cap \overline{A}(S)$ if and only if for every $r \in B$

$r \in p\uparrow^{\vee} \cap \overline{A}(S) \longleftrightarrow r \in q\uparrow^{\wedge} \cap \overline{A}(S)$ if and only if for every $x \in S$

$\overline{A}(x) \geq_{\vee} p \longleftrightarrow \overline{A}(x) \geq_{\wedge} q$ if and only if for every $x \in S$

$x \in A_p^{\vee} \longleftrightarrow x \in A_q^{\wedge}$ if and only if for every $x \in S$

$\overline{A}_p^{\vee}(x) = 1 \longleftrightarrow \overline{A}_q^{\wedge}(x) = 1$ if and only if

$v_p^{\vee} = v_q^{\wedge}$. □

Corollary 1. *Let* $\overline{A} : S \to B$ *be a bisemilattice valued fuzzy set and* $p, q \in B$. *If* $p\uparrow^{\vee} = q\uparrow^{\wedge}$ *then* $v_p^{\vee} = v_q^{\wedge}$. □

If $V_{\overline{A}}$ is the binary block-code which corresponds to B-fuzzy set $\overline{A}$, then obviously $|V_{\overline{A}}| \leq |\mathcal{F}^{\vee} \cup \mathcal{F}^{\wedge}|$. By III and IV, a bisemilattice valued fuzzy set $\overline{A}$ has the foregoing number of different levels (code-words of $V_{\overline{A}}$), if all meet-irreducible element from the bisemilattice B are in $\overline{A}(S)$. It is easy to construct a B-fuzzy set which fulfills this condition: $\overline{A}$ could be the mapping of the set S *onto* B. However, our aim is to obtain a fuzzy set whose co-domain is a minimal extension of the collection of meet-irreducible elements of B. Hence, the domain of such fuzzy set would be the smallest possible, in other words the length of the corresponding code would be minimal.

In the following we present an algorithm for the construction of a B-fuzzy set, B being a fixed finite bisemilattice, with maximal number $|\mathcal{F}^{\vee} \cup \mathcal{F}^{\wedge}|$ of levels (code-words) and minimal domain.

ALGORITHM

Let $(B, \vee, \wedge)$ be a given fixed finite bisemilattice and $J = \{a_1, \ldots, a_n\}$ the set of its meet-irreducible elements.

1. Further, let $S = \{1, \ldots, n\}$ (note that $n = |J|$) and $\overline{A}$ a bijection from S to J. Obviously, $\overline{A}$ is a B-fuzzy set on S with distinct levels (by **III** and **IV**). If the number of levels is maximal, i.e., if $|V_{\overline{A}}| = |\mathcal{F}^{\vee} \cup \mathcal{F}^{\wedge}|$, then the procedure is over.

2. If the number of levels of $\overline{A}$ is less than $|\mathcal{F}^\vee \cup \mathcal{F}^\wedge|$, then we construct a new B-fuzzy set in the following way. Let

$$(\mathcal{F}^\vee)' := \mathcal{F}^\vee \setminus (\mathcal{F}^\vee \cap \mathcal{F}^\wedge), \quad (\mathcal{F}^\wedge)' := \mathcal{F}^\wedge \setminus (\mathcal{F}^\vee \cap \mathcal{F}^\wedge).$$

Further, divide each ideal from $(\mathcal{F}^\vee)'$ and from $(\mathcal{F}^\wedge)'$ in two parts; the first with meet-irreducible elements or the empty set, and the second with all other elements of that ideal or the empty set, as follows:

$$(\mathcal{F}^\vee)'_J := \{(c_J, c) \mid c_J = f \cap J, c_J \cup c = f, \text{ for some } f \in (\mathcal{F}^\vee)'\};$$

$$(\mathcal{F}^\wedge)'_J := \{(d_J, d) \mid d_J = g \cap J, d_J \cup d = g, \text{ for some } g \in (\mathcal{F}^\wedge)'\}.$$

3. Now we form a table, first row of which consists of pairs from $(\mathcal{F}^\vee)'_J$.

4. To form the second row, we look for the members of $(\mathcal{F}^\wedge)'_J$ which have the same first part as some elements from the first row, and put them in the same column. In all other columns of the second row we put the sign $*$.

Hence, after two rows, columns of the table are either of the form (c_J, c), (c_J, d) and $*$, or they have elements (c_J, c) and $*$. Note that elements from $(\mathcal{F}^\wedge)'_J$ in the second row of the table are uniquely determined, since by proposition **I**, distinct filters could not have the same set of meet-irreducible elements.

Let s_{II} be the number of signs $*$ in the second row. Then the number of different levels for the fuzzy set $\overline{A}$, which is a bijection from S to J, is by the above construction

$$|V_{\overline{A}}| = |\mathcal{F}^\vee \cap \mathcal{F}^\wedge| + 2s_{II} + (|(\mathcal{F}^\vee)'_J| - s_{II}). \tag{2}$$

By the assumption in **1.**, this number is less than $|\mathcal{F}^\vee \cup \mathcal{F}^\wedge|$, and we proceed to the following step.

5. The third row of the table repeats signs $*$ in the same columns as in the second row, and in the remaining columns we put symmetric differences (Δ) of the second coordinate of pairs situated above (in the same column): (c_I, c) and (c_I, d) are followed by $c\Delta d$. If more than one column have equal sets in this row, then we underline all but one of them. Denote by K the set-union of singletons of the third row. Each column having an element in K is also underlined. Let t_{III} be the number of underlined columns in the third row.

6. If $|J \cup K| = m$ and $S_1 = \{1, \ldots, m\}$, then a bijection $\overline{A}_1 : S_1 \to J_1$, $J_1 = J \cup K$, is a fuzzy set with distinct levels, and the number of these levels and code-words of $V_{\overline{A}}$ is by the construction $|V_{\overline{A}_1}| = |V_{\overline{A}}| + t_{III}$. If this number is equal to $|\mathcal{F}^{\vee} \cup \mathcal{F}^{\wedge}|$, the procedure is over, otherwise we proceed to the following step.

7. The row four of the table has signs $*$ in all columns in which the third row has, and also in columns which were previously underlined. The remaining columns are equal as in the third row, and by the construction they all have two or more elements. Denote by L the union of two-element sets (if any) in this row and underline all columns of the row containing some element from L. Now, add to J_1 any element a with maximal number of appearances in L, and underline all columns of the row containing a; the union of non-underlined two-columns denote by L_1. If there are elements in L_1 with more than one appearance, repeat the procedure. Otherwise, add one element from each non-underlined column to $J_1 \cup \{a\}$.

The number of levels (code-words) increase, and if it is not maximal, then the procedure from **7.** applies to the last row.

This process continues until maximal number of code-words is obtained.

8. In the co-domain of every fuzzy set obtained by the foregoing procedure, look for maximal elements of the semilattice $(B, \wedge)$ which fulfill the assumptions of proposition **III** and which are not meet irreducible in $(B, \vee)$ (by **IV**). Then eliminate one of them, provided that this elimination does not generate equality of distinct filters used in the algorithm.

Any of fuzzy sets obtained in this way is a result of the algorithm.

Example 2.

Bisemilattice $(B, \vee, \wedge)$ is represented by two diagrams in Fig. 2.

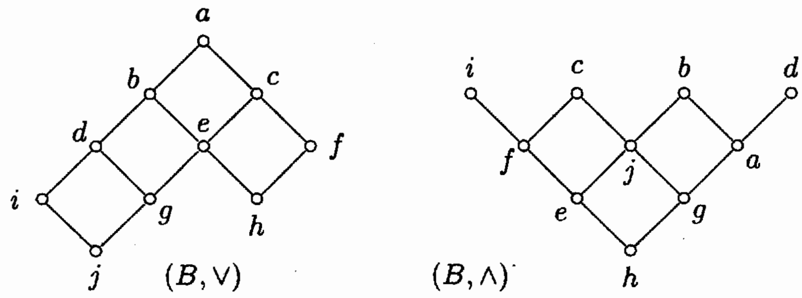

Figure 2

Applying the above algorithm, we get the following.

$\mathcal{F}^{\vee} = \{a\uparrow^{\vee}, b\uparrow^{\vee}, c\uparrow^{\vee}, \dots, j\uparrow^{\vee}\}$;

$\mathcal{F}^{\wedge} = \{a\uparrow^{\wedge}, b\uparrow^{\wedge}, c\uparrow^{\wedge}, \dots, j\uparrow^{\wedge}\}$.

All the above filters are sets and it is easy to see that only two of them are equal, one in each collection. Namely, $d\uparrow^{\vee} = \{a, b, d\} = a\uparrow^{\wedge}$. Hence,

$|\mathcal{F}^{\vee} \cap \mathcal{F}^{\wedge}| = 1$.

$(\mathcal{F}^{\vee})' = \{\{a\}, \{a,b\}, \{a,c\}, \{a,b,c,e\}, \{a,c,f\}, \{a,b,c,d,e,g\}, \{a,b,c,e,f,h\}, \{a,b,c,d,e,g,i,j\}, \{a,b,d,i\}\}$;

$(\mathcal{F}^{\wedge})' = \{\{b\}, \{c\}, \{d\}, \{b,c,e,f,i,j\}, \{c,f,i\}, \{a,b,c,d,j,g\}, \{a,b,c,d,e,f,g,h,i,j\}, \{b,c,j\}, \{i\}\}$;

$J = \{b, c, d, f, i\}$;

$|\mathcal{F}^{\vee} \cup \mathcal{F}^{\wedge}| = 19$.

Now, the mapping

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ b & c & d & f & i \end{pmatrix}$$

is a B-fuzzy set with distinct levels. It is easy to check that the number of these levels, i.e., the code-words of the corresponding block-code is given by $|V_{\overline{A}}| = 16$.

The table described by the algorithm is given in the following.

I	$(\emptyset, \{a\})$	$(\{b\}, \{a\})$	$(\{c\}, \{a\})$	$(\{b,c\}, \{a,e\})$	$(\{c,f\}, \{a\})$	$(\{b,c,d\}, \{a,e,g\})$
II	*	$(\{b\}, \emptyset)$	*	$(\{b,c\}, \{j\})$	*	$(\{b,c,d\}, \{a,g,j\})$
III	*	$\underline{\{a\}}$	*	$\underline{\{a,e,j\}}$	*	$\{e,j\}$
IV	*	*	*	*	*	$\underline{\{e,j\}}$

	$(\{b,c,f\},\{a,e,h\})$	$(\{b,c,d,i\},\{a,e,g,j\})$	$(\{b,d,i\},\{a\})$	
	*	*	*	$\lvert V_{\overline{A}}\rvert = 16$
	*	*	*	$\lvert V_{\overline{A}_1}\rvert = 18$
	*	*	*	$\lvert V_{\overline{A}_2^{(i)}}\rvert = 19$

The number of code-words $|V_{\overline{A}}| = 16$, can be checked by formula (2).

Since there is one singleton $(\{a\})$ in the third row, i.e., $K = \{a\}$, it follows that $J_1 = J \cup K = \{a, b, c, d, f, i\}$, and the fuzzy set

$$\overline{A}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & f & i \end{pmatrix}$$

has 18 distinct levels, code-words of $V_{\overline{A}_1}$ (step **6.** of the algorithm).

Finally, there is one two-element set $\{e, j\}$ in the fourth row. By step **7.**, there are two possibilities for a fuzzy set with 19 levels:

$$\overline{A}_2' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & b & c & d & e & f & i \end{pmatrix} \quad \text{and} \quad \overline{A}_2'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & b & c & d & f & i & j \end{pmatrix}.$$

Maximal number of code-words is 19. Since there is no maximal element in $(B, \wedge)$ fulfilling assumption of proposition **III**, the procedure is over. □

Maximal number of code-words is always reached by the algorithm. Indeed, this number is obtained at least when all elements from B are used. We prove that the obtained code has minimal length.

Proposition 2. *For a fixed finite bisemilattice B, the code $V_{\overline{A}}$ obtained by the above algorithm has minimal length in the set of all maximal codes corresponding to B-fuzzy sets.*

Proof. The co-domain $\overline{A}(S)$ of a B-fuzzy set $\overline{A} : S \to B$ with distinct levels contains:

- all meet-irreducible elements from B with exception of one maximal if it satisfies particular conditions (by proposition **III**);

- for any $p\uparrow^{\vee}$ and $q\uparrow^{\wedge}$, elements which are not meet-irreducible, such that $p\uparrow^{\vee} \cap \overline{A}(S) \neq q\uparrow^{\wedge} \cap \overline{A}(S)$ (by Lemma 2).

The co-domain of a fuzzy set obtained by the algorithm is by the construction contained in a co-domain of every fuzzy set which satisfies the above conditions. Since it is a bijection, its domain and thus the length of the corresponding code is minimal □

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Received by the editors September 14, 1998.