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CONGRUENCES OF n-GROUP AND OF ASSOCIATED HOSSZÚ-GLUSKIN ALGEBRAS

Janez Ušan
Institute of Mathematics,
University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad,
Yugoslavia

Abstract

For every n-group (Q, A), $n \geq 3$, there is an algebra $(Q, \{\cdot, \varphi, b\})$ [of the type $\langle 2,1,0\rangle$] such that the following statements hold: 1° $(Q, \cdot)$ is a group; 2° $\varphi \in Aut(Q, \cdot)$; 3° $\varphi(b) = b$; 4° for every $x \in Q$ $\varphi(x) \cdot b = b \cdot x$; and 5° for every $x_1^n \in Q$ $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$ [:Hosszú-Gluskin Theorem [2-3]]. We say that an algebra $(Q, \{\cdot, \varphi, b\})$ is a Hosszú-Gluskin algebra of order n $(n \geq 3)$ [briefly: nHG-algebra] iff the statements 1°-4° hold. In addition, we say that an nHG-algebra $(Q, \{\cdot, \varphi, b\})$ is associated to an n-group (Q, A) iff 5° holds. [in [10], all nHG-algebras **associated** to the given n-group are described]. One of the main results of the paper is the following proposition: Let$n \geq 3$, and let (Q, A) be an n-group. Further on, let $(Q, \{\cdot, \varphi, b\})$ be its **arbitrary** associated nHG-algebra. Then $Con(Q, A) = Con(Q, \cdot) \cap Con(Q, \varphi)$. In addition, in the present paper we prove that the congruence lattice of an n-group (Q, A) is a **sublattice** of the congruence lattice of the group $(Q, \cdot)$ and that it is isomorphic with the lattice of normal subgroups $(H, \cdot)$ of the group $(Q, \cdot)$ for which $\varphi(H) = H$. [In [4], Monk and Sioson described the congruence lattice of the n-group $(n \geq 3)$, up to an isomorphism, in the scope of the Post covering group. (:Remark 5.3).] In this paper, we also prove the following proposition: Let $n \geq 3$ and let (Q, A) be an n-group. Further on, let θ be an arbitrary element of the set $Con(Q, A)$.

Then, **for every** $C_t \in Q/\theta$ **there is** an nHG-algebra $(Q, \{\cdot, \varphi, b\})$ associated to the n-group (Q, A) such that the following statements hold: (i) $(C_t, \cdot) \triangleleft (Q, \cdot)$; (ii) (C_t, φ) is a 1-groupoid; and (iii) (C_t, A) is an n-subgroup of the n-group (Q, A) iff $b \in C_t$.

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1. Preliminaries

1.1. About the expression a_p^q

Let $p \in N$, $q \in N \cup \{0\}$ and let a be a mapping of the set $\{i | i \in N \wedge i \geq p \wedge i \leq q\}$ into the set $S; \emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence}(= \emptyset); & p > q. \end{cases}$$

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S .... And if $p \leq q$, we usually write: $a_p^q \in S$.

If a_p^q is a sequence over a set S, $p \leq q$ and the equalities $a_p = \ldots = a_q = b (\in S)$ are satisfied, then

$$a_p^q \textit{ is denoted by } \overset{q-p+1}{b} .$$

1.2. About n-groups

1.2.1. Definitions: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then: (a) we say that (Q, A) is an n-semigroup iff for every $i, j \in \{1, \ldots, n\}$, $i < j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[: $\langle i,j\rangle$-associative law]; (b) we say that (Q,A) is an n-quasigroup iff for every $i \in \{1,\ldots,n\}$ and for every $a_1^n \in Q$ there is **exactly one** *$x_i \in Q$ such that the following equality holds*

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n; \text{ and}$$

(c) we say that (Q,A) is a Dörnte n-group [briefly: n-group] iff (Q,A) is an n-semigroup and an n-quasigroup as well.

A notion of an n-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

1.2.2. Definition [8]: *Let $n \geq 2$ and let (Q,A) be an n-groupoid. Further on, let* e *be an mapping of the set Q^{n-2} into the set Q. Let also $\{i,j\} \subseteq \{1,\ldots,n\}$ and $i < j$. Then:* e *is an $\{i,j\}$-neutral operation of the n-groupoid (Q,A) iff the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q) \quad (A(a_1^{i-1}, \mathsf{e}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x \\ \wedge \quad A(a_1^{i-1}, x, a_i^{j-2}, \mathsf{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x).$$ [1]

1.2.3. Proposition [8]: *Let $n \geq 2$, $\{i,j\} \subseteq \{1,\ldots,n\}$ and $i < j$. Then in every n-groupoid there is* **at most one** *$\{i,j\}$-neutral operation.*

1.2.4. Proposition [8]: *In every n-group, $n \geq 2$, there is a $\{1,n\}$-neutral operation.*[2]

1.2.5. Proposition [10]: *Let (Q,A) be an n-group,* e *its $\{1,n\}$-neutral operation and $n \geq 3$. Then for every sequence a_1^{n-2} over Q and for every $i \in \{1,\ldots,n-2\}$ there is exactly one $x_i \in Q$ such that the equality*

$$\mathsf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2}$$

holds.

1.2.6. Proposition [9]: *Let $n \geq 2$, and let (Q,A) be an n-group,* e *its $\{1,n\}$-neutral operation and* E *a $\{1,2n-1\}$-neutral operation of a $(2n-1)$-group $(Q,\overset{2}{A})$, where $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$. Further on, let f be an $(n-1)$-ary operation in Q defined in the following way*

$$f(a_1^{n-2}, a) \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}).$$

[1] For $n = 2$, $\mathsf{e}(a_1^{n-2})$ $[= \mathsf{e}(\emptyset)] = e \in Q$ is a neutral element of the groupoid (Q,A).

[2] There are n-groups without $\{i,j\}$-neutral operations with $\{i,j\} \neq \{1,n\}$ [:[11]]. In [11], n-groups with $\{i,j\}$-neutral operations, for $\{i,j\} \neq \{1,n\}$ are described.

Then, also the following laws hold in the algebra $(Q,\{A,f,\mathbf{e}\})$ *[of the type* $\langle n,n-1,n-2\rangle$*]*

$$A(f(a_1^{n-2},a),a_1^{n-2},a)=\mathbf{e}(a_1^{n-2}) \text{ and}$$
$$A(a,a_1^{n-2},f(a_1^{n-2},a))=\mathbf{e}(a_1^{n-2}).^3$$

1.2.7. Remark: As well as Proposition 1.2.4 and Proposition 1.2.6, for $n\geq 2$. e. g. the following proposition holds [13]: *If the following laws hold in the algebra* $(Q,\{A,^{-1},\mathbf{e}\})$ *of the type* $\langle n,n-1,n-2\rangle$

$$A(A(x_1^n),x_{n+1}^{2n-1})=A(x_1,A(x_2^{n+1}),x_{n+2}^{2n-1}),$$
$$A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))=x \text{ and}$$
$$A(a,a_1^{n-2},(a_1^{n-2},a)^{-1})=\mathbf{e}(a_1^{n-2}),$$

then (Q,A) *is an* n*-group. For* $n=2$ *this is the well known characterizations of* n*-groups.*

1.3. On Hosszú-Gluskin algebras

1.3.1. Proposition (Hosszú-Gluskin Theorem) [2-3]: *For every* n*-group* (Q,A), $n\geq 3$, *there is an algebra* $(Q,\{\cdot,\varphi,b\})$ *such that the following statements hold: 1° $(Q,\cdot)$ is a group; 2° $\varphi\in Aut(Q,\cdot)$; 3° $\varphi(b)=b$; 4° for every $x\in Q$, $\varphi^{n-1}(x)\cdot b=b\cdot x$; and 5° for every $x_1^n\in Q$, $A(x_1^n)=x_1\cdot\varphi(x_2)\cdot\ldots\cdot\varphi^{n-1}(x_n)\cdot b$.*

1.3.2. Definitions [10]: *We say that an algebra* $(Q,\{\cdot,\varphi,b\})$ *is a Hosszú-Gluskin algebra of order* n *(*$n\geq 3$*) [briefly: nHG-algebra] iff 1°-4° from 1.3.1 holds. In addition, we say that an nHG-algebra* $(Q,\{\cdot,\varphi,b\})$ *is* **associated** *to the* n*-group* (Q,A) *iff 5° from 1.3.1 hold.*

1.3.3. Proposition [10]: *Let* $n\geq 3$, *let* (Q,A) *be an* n*-group, and* $\mathbf{e}$ *its* $\{1,n\}$*-neutral operation. Further on, let* c_1^{n-2} *be an arbitrary sequence over* Q *and let for every* $x,y\in Q$

$$B_{(c_1^{n-2})}(x,y)\stackrel{def}{=}A(x,c_1^{n-2},y); \tag{1}$$

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[3] For $n=2$, f is the inversing operation in the group (Q,A). In addition, for $n=2$: $a^{-1}[=f(a)]=\mathsf{E}(a)$; $a_1^{n-2}=\emptyset$, 1.1. In some papers, the authors writes $(a_1^{n-2},a)^{-1}$ instead of $f(a_1^{n-2},a)$.

(2) $$\varphi_{(c_1^{n-2})}(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \text{ and}$$

(3) $$b_{(c_1^{n-2})} \overset{def}{=} A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n})^4.$$

Then, the following statements hold

(i) $(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\})$ *is an nHG-algebra associated to the n-group* (Q, A)*; and*

(ii) $\mathcal{C}_A \overset{def}{=} \{(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) | \, c_1^{n-2}$ *is a* **sequence** *over* $Q\}$ *is the set of all nHG-algebras* **associated** *to the n-group* (Q, A).

1.4. On congruences in an m-groupoid

1.4.1. Definition: *Let* (Q, F) *be an m-groupoid and* $m \in \mathbf{N}$. *Let also* θ *be an equivalence relation in the set* Q. *Then,* θ *is a* **congruence relation** *on the m-groupoid* (Q, F) *iff the following holds*

$$(\forall a_j \in Q)_1^m (\forall b_j \in Q)_1^m \; ((\bigwedge_{i=1}^{m} a_i \theta b_i) \Rightarrow F(a_1^m) \theta F(b_1^m)).$$

1.4.2. Proposition: θ *is a* **congruence** *on an m-groupoid* (Q, F) *iff the following holds*

$$(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{m-1}$$
$$(\bigwedge_{i=1}^{m} (a\theta b \Rightarrow F(c_1^{i-1}, a, c_i^{m-1}) \theta F(c_1^{i-1}, a, c_i^{m-1}))).$$

1.4.3. Definition: *A congruence relation* θ *on an m-groupoid* (Q, F) *is said to be* **normal** *iff the following holds*

$$(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{m-1}$$
$$(\bigwedge_{i=1}^{m} (F(c_1^{i-1}, a, c_i^{m-1}) \theta F(c_1^{i-1}, a, c_i^{m-1}) \Rightarrow a\theta b)).^5$$

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[4] $A(\mathbf{e}(c_1^{n-2}), \ldots, \mathbf{e}(c_1^{n-2}))$.

[5] Normal congruences on quasigroups ($m = 2$) are described e. g. in [5] [p.54].

1.4.4. Proposition [12]: *Let $n \geq 2$, let (Q, A) be an n-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation [:1.2.4], f its inversing operation [:1.2.6], and θ a congruence on the groupoid (Q, A) [:$\theta \in Con(Q, A)$]. Then: (a) θ is a normal congruence of the n-groupoid (Q, A); (b) for $n \geq 3$ θ is a normal congruence of the $(n-2)$-groupoid $(Q, \mathbf{e})$; and (c) θ is a congruence of the $(n-1)$-groupoid (Q, f).*[6]

2. Construction of a lattice on a given nHG-algebra

2.1. Proposition: *Let $(Q, \{\cdot, \varphi, b\})$ be an nHG-algebra [:1.3.2]. Further on, let*

$$L \stackrel{def}{=} \{H | (H, \cdot) \lhd (Q, \cdot)\}^7 \text{ and} \tag{1}$$

$$\hat{L} \stackrel{def}{=} \{H | (H, \cdot) \lhd (Q, \cdot) \wedge \varphi(H) = H\}^8. \tag{2}$$

Then $(\hat{L}, \odot, \cap)$[9] is a sublattice of the (modular) lattice $(L, \odot, \cap)$.

Proof. 1) $(L, \odot, \cap)$ is the well known modular lattice of normal subgroups of the group $(Q, \cdot)$.

2) Since $\varphi \in Aut(Q, \cdot)$ and by the definition of the operation $\odot$ we conclude that $(\hat{L}, \odot)$ is a groupoid.

3) By the definition of the set $\hat{L}$ [:(2)] and by the definition of the set $\varphi(H)$ [footnote 8)] we conclude that for every $x \in Q$ and for every $H_1, H_2 \in \hat{L}$ the following series of equivalences hold

$$\begin{aligned} \varphi(x) \in \varphi(H_1 \cap H_2) &\Leftrightarrow x \in H_1 \cap H_2 \\ &\Leftrightarrow x \in H_1 \wedge x \in H_2 \\ &\Leftrightarrow \varphi(x) \in \varphi(H_1) \wedge \varphi(x) \in \varphi(H_2) \\ &\Leftrightarrow \varphi(x) \in H_1 \wedge \varphi(x) \in H_2 \\ &\Leftrightarrow \varphi(x) \in H_1 \cap H_2, \end{aligned}$$

[6] For $n = 2$, θ is a normal congruence of the 1-groupoid (Q, f) [:$(Q, ^{-1})$].

[7] $(H, \cdot) \lhd (Q, \cdot)$: $(H, \cdot)$ is a normal subgroup of the group $(Q, \cdot)$.

[8] $\varphi(H) \stackrel{def}{=} \{\varphi(x) | x \in H\}$ [:$\varphi(x) \in \varphi(H) \Leftrightarrow x \in H; x \in Q$].

[9] $H_1 \odot H_2 \stackrel{def}{=} \{x | x = h_1 \cdot h_2 \wedge h_1 \in H_1 \wedge h_2 \wedge H_2\}$; $H_1, H_2 \in P(Q)$. $H_1 \cap H_2$ is the intersection of the sets H_1 and H_2.

i. e., that the following equivalence

$$\varphi(x) \in \varphi(H_1 \cap H_2) \Leftrightarrow \varphi(x) \in H_1 \cap H_2,$$

whence, since φ is a permutation of the set Q, we conclude that the set $\hat{L}$ is closed also under the operation $\cap$.

4) Finally, by the Propositions from 1)-3), we conclude that $(\hat{L}, \odot, \cap)$ is a sublattice of the modular lattice $(L, \odot, \cap)$.

2.2. Example: Let $(\{1,2,3,4\}, \cdot)$ be the Klein's group: Tab. 1.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Tab. 1.

Further on, let φ be the permutation of the set $\{1,2,3,4\}$ defined in the following way

$$\varphi \overset{def}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

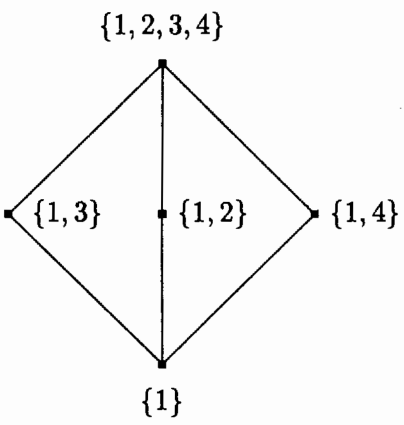
Then, $(\{1,2,3,4\}, \{\cdot, \varphi, 2\})$ is a 3HG-algebra. In addition, the following holds:

$$L = \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3,4\}\} \text{ [:(1) from 2.1] and}$$

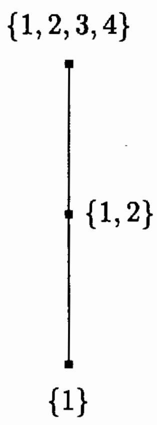
$$\hat{L} = \{\{1\}, \{1,2\}, \{1,2,3,4\}\} \text{ [:(2) from 2.1];}$$

$\varphi(\{1,3\}) = \{1,4\} \neq \{1,3\}$, $\varphi(\{1,4\}) = \{1,3\} \neq \{1,4\}$. Lattices $(L, \odot, \cap)$ and $(\hat{L}, \odot, \cap)$ [:2.1] are represented in Diag. 1 and Diag. 2.

2.3. Remark: *If $(Q, \{\cdot, \varphi, b\})$ is an nHG-algebra and φ an inner automorphism of the group $(Q, \cdot)$, then $(\hat{L}, \odot, \cap) = (L, \odot, \cap)$. However, there are nHG-algebras $(Q, \{\cdot, \varphi, b\})$ such that φ is not an inner automorphism of the group $(Q, \cdot)$ and $(\hat{L}, \odot, \cap) = (L, \odot, \cap)$. E. g.: Let $(Q, \cdot)$ be a commutative group, $^{-1}$ an inversing operation in $(Q, \cdot)$ and let there is at least one $x \in Q$ such that $x^{-1} \neq x$. Further on, let $\varphi = ^{-1}$, $b = \mathrm{e}$, where e is the neutral element of the group $(Q, \cdot)$. Then $(Q, \{\cdot, \varphi, b\})$ is a 3HG-algebra and $\hat{L} = L$ [:2.1].*



Diag. 1



Diag. 2

3. An auxiliary proposition

3.1. Proposition: *Let* $(Q, \{\cdot, \varphi, b\})$ *be an nHG-algebra and let* $^{-1}$ *be an inversing operation in the group* $(Q, \cdot)$. *Further on, let* $(H, \cdot) \lhd (Q, \cdot)$. *Then the following statements are equivalent:*

(i) $(\forall x \in Q)(\forall y \in Q)(x \cdot y^{-1} \in H \Rightarrow \varphi(x \cdot y^{-1}) \in H)$;

(ii) $(\forall x \in Q)(\forall y \in Q)(x \cdot y^{-1} \in H \Leftrightarrow \varphi(x \cdot y^{-1}) \in H)$; *and*

(iii) $\varphi(H) = H$.

Proof.

(i)$\Leftrightarrow$ (ii):

Let (i) holds. Then, since $(Q, \{\cdot, \varphi, b\})$ is an nHG-algebra [:1.3.2], and since $(H, \cdot)$ is a normal subgroup of the group $(Q, \cdot)$ and since $^{-1}$ is an inversing operation in $(Q, \cdot)$, we conclude that for every $x, y \in Q$ the following series of implications holds

$$\varphi(x \cdot y^{-1}) \in H \text{ [10]} \Rightarrow \varphi^{n-1}(x \cdot y^{-1}) \in H \Rightarrow$$
$$\varphi^{n-1}((x \cdot b) \cdot (y \cdot b)^{-1}) \in H \Rightarrow \varphi^{n-1}(x \cdot b) \cdot \varphi^{n-1}((y \cdot b)^{-1}) \in H \Rightarrow$$
$$\varphi^{n-1}((x \cdot b) \cdot (\varphi^{n-1}(y \cdot b))^{-1} \in H \Rightarrow (b \cdot x) \cdot (b \cdot y)^{-1} \in H \Rightarrow$$
$$b \cdot (x \cdot y^{-1}) \cdot b^{-1} \in H \Rightarrow b \cdot (x \cdot y^{-1}) \cdot b^{-1} \in bHb^{-1} \Rightarrow$$
$$x \cdot y^{-1} \in H,$$

i. e., that the following statement holds

(1) $$(\forall x \in Q)(\forall y \in Q)(\varphi(x \cdot y^{-1}) \in H \Rightarrow x \cdot y^{-1} \in H).$$

Since the conjunction of the statements (1) and (i) is equivalent with the statement (ii), the equivalence (i) $\Leftrightarrow$ (ii) holds.

(ii)$\Rightarrow$ (iii):

Let (ii) holds. Then, for every $x \in Q$ the following equivalence holds

$$x \in H \Leftrightarrow \varphi(x) \in H,$$

whence, since for every $x \in Q$

$$\varphi(x) \in \varphi(H) \Leftrightarrow x \in H,$$

we conclude that for every $x \in Q$ the following equivalence holds

$$\varphi(x) \in \varphi(H) \Leftrightarrow \varphi(x) \in H.$$

Whence, since φ is a permutation of the set Q, we conclude that

$$\varphi(H) = H.$$

(iii)$\Rightarrow$ (ii):

Let (iii) holds. Then, for every $x, y \in Q$ the following equivalence holds

$$\varphi(x \cdot y^{-1}) \in \varphi(H) \Leftrightarrow \varphi(x \cdot y^{-1}) \in H,$$

whence, since for every $x, y \in Q$

$$\varphi(x \cdot y^{-1}) \in \varphi(H) \overset{def}{\Leftrightarrow} x \cdot y^{-1} \in H,$$

we conclude that the statement (ii) holds.

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[10] $\varphi(x \cdot y^{-1}) \in H \Leftrightarrow \varphi(x) \cdot (\varphi(y))^{-1} \in H$.

4. On the set of all congruences of the given n-group ($n \geq 3$)

4.1. Theorem: *Let $n \geq 3$, let (Q, A) be an n-group and let $(Q, \{\cdot, \varphi, b\})$ be an* **arbitrary** *nHG-algebra associated to the n-group (Q, A). Then, the following equality holds*

$$Con(Q, A) = Con(Q, \cdot) \cap Con(Q, \varphi).^{11}$$

Proof. 1) $\Rightarrow$:

Let $\mathbf{e}$ be a $\{1, n\}$-neutral operation of an n-group (Q, A) [:1.2.4] and let c_1^{n-2} be an arbitrary sequence over the set Q. Further on, for every $x, y \in Q$ the following hold

$$x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y), \tag{1}$$

$$\varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \text{ and} \tag{2}$$

$$b \overset{def}{=} A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}). \tag{3}$$

Then $(Q, \{\cdot, \varphi, b\})$ is an nHG-algebra associated to the n-group (Q, A) [:1.3. 3]. In addition, for every $(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A$ there is a sequence c_1^{n-2} over Q such that for all $x, y \in Q$ (1)-(3) [:1.3.3]. Further on, if $\theta \in Con(Q, A)$ [:1.4], since (1) and (2) hold for all $x, y \in Q$, we conclude that for all $x, y, \bar{x}, \bar{y} \in Q$ the following sequence of implications holds

$$\begin{aligned}
x\theta\bar{x} &\Rightarrow A(x, c_1^{n-2}, y)\theta A(\bar{x}, c_1^{n-2}, y) \\
&\Rightarrow x \cdot y\theta\bar{x} \cdot y \\
y\theta\bar{y} &\Rightarrow A(x, c_1^{n-2}, y)\theta A(x, c_1^{n-2}, \bar{y}) \\
&\Rightarrow x \cdot y\theta x \cdot \bar{y} \\
x\theta\bar{x} &\Rightarrow A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})\theta A(\mathbf{e}(c_1^{n-2}), \bar{x}, c_1^{n-2}) \\
&\Rightarrow \varphi(x)\theta\varphi(\bar{x}),
\end{aligned}$$

[11] $\theta \in Con(Q, A) \Rightarrow \theta \in Con(Q, \mathbf{e}) \wedge \theta \in Con(Q, ^{-1})$ [:1.4.4].

whence we conclude that for an arbitrary $(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A$ and for arbitrary $\theta \in P(Q^2)$, the following implication holds

$$\theta \in Con(Q, A) \Rightarrow \theta \in Con(Q, \cdot) \wedge \theta \in Con(Q, \varphi).$$

2) $\Leftarrow$:

Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary algebra from the set $\mathcal{C}_A$ [:1.3.3]. Further on, let θ be an arbitrary element of the set $P(Q^2)$ such that the following conjunction holds

$$\theta \in Con(Q, \cdot) \wedge \theta \in Con(Q, \varphi).$$

Since $\theta \in Con(Q, \cdot)$, the following statement holds:

(a) for every $i \in \{1, \ldots, m\}$, $m \in \mathbf{N} \setminus \{1\}$, for every sequence a_1^m over the set Q and for all $x, \bar{x} \in Q$ the following implication holds

$$x\theta\bar{x} \Rightarrow (\prod_{j=1}^{i-1} a_j) \cdot x \cdot (\prod_{j=i}^{m-1} a_j)\theta(\prod_{j=1}^{i-1} a_j) \cdot \bar{x} \cdot (\prod_{j=i}^{m-1} a_j)^{12}.$$

Since $\theta \in Con(Q, \varphi)$, the following statement holds:

(b) for every $t \in \mathbf{N} \cup \{0\}$ and for all $x, \bar{x} \in Q$ the following implication holds

$$x\theta\bar{x} \Rightarrow \varphi^t(x)\theta\varphi^t(\bar{x}).$$

Finally, by (a), (b) and by the assumption that $(Q, \{\cdot, \varphi, b\}) \in \mathcal{C}_A$ [:1.3.3], we conclude that for every $i \in \{1, \ldots, n\}$, for every $x_1^n \in Q$ and for every $\bar{x}_1^n \in Q$ the following series of implications holds

$$\begin{aligned} x_i\theta\bar{x}_i \; &\Rightarrow \; \varphi^{i-1}(x_i)\theta\varphi^{i-1}(\bar{x}_i) \\ &\Rightarrow \; (\prod_{j=1}^{i-1} \varphi^{j-1}(x_j)) \cdot \varphi^{i-1}(x_i) \cdot (\prod_{j=i+1}^{n} \varphi^{j-1}(x_j)) \cdot b \,\theta \\ & \; (\prod_{j=1}^{i-1} \varphi^{j-1}(x_j)) \cdot \varphi^{i-1}(\bar{x}_i) \cdot (\prod_{j=i+1}^{n} \varphi^{j-1}(x_j)) \cdot b \\ &\Rightarrow \; A(x_1^{i-1}, x_i, x_{i+1}^n)\theta A(x_1^{i-1}, \bar{x}_i, x_{i+1}^n). \end{aligned}$$

[12] $\prod_{j=p}^{p-1} a_j \overset{def}{=} \mathbf{e}$, where $\mathbf{e}$ is the neutral element of the group $(Q, \cdot)$, and $p \in \mathbf{N}$.

4.2. Remark: *Let* $(\{1,2,3,4\},\{\cdot,\varphi,2\})$ *3HG-algebra from Example 2.2. Equivalence relations* $\theta_1 - \theta_5$ *in* $\{1,2,3,4\}$ *defined as follows, belong to the set* $Con(\{1,2,3,4\},\cdot)$*:*

$$\{1,2,3,4\}/\theta_1 \overset{def}{=} \{\{1,3\},\{2,4\}\};$$

$$\{1,2,3,4\}/\theta_2 \overset{def}{=} \{\{1,4\},\{2,3\}\};$$

$$\{1,2,3,4\}/\theta_3 \overset{def}{=} \{\{1,2\},\{3,4\}\};$$

$$\{1,2,3,4\}/\theta_4 \overset{def}{=} \{\{1\},\{2\},\{3\},\{4\}\} \text{ and}$$

$$\{1,2,3,4\}/\theta_5 \overset{def}{=} \{\{1,2,3,4\}\}.$$

As well as $\theta_3 - \theta_5$*, equivalence relations* θ_6 *and* θ_7 *in* $\{1,2,3,4\}$ *defined as follows, belong to the set* $Con(\{1,2,3,4\},\varphi)$*:*

$$\{1,2,3,4\}/\theta_6 \overset{def}{=} \{\{1\},\{2,3,4\}\} \text{ and}$$

$$\{1,2,3,4\}/\theta_7 \overset{def}{=} \{\{2\},\{1,3,4\}\}.$$

The set $Con(Q,A)$*, where*

$$A(x_1^3) \overset{def}{=} x_1 \cdot \varphi(x_2) \cdot x_3 \cdot 2 \text{ [:1.3]},$$

by Theorem 4.1 is the set $\{\theta_3,\theta_4,\theta_5\}$.

In the Theory of groups ($n = 2$) the following proposition is well known:

4.3. Proposition: *Let* $(Q,\cdot)$ *be a group and let* $^{-1}$ *be its inversing operation. Further on, let*

$$L \overset{def}{=} \{H|(H,\cdot) \lhd (Q,\cdot)\} \text{ [:(1) from 2.1]}.$$

Then, there is **exactly one bijection** F *of the set* $Con(Q,\cdot)$ *onto the set* L *such that* **for every** $\theta \in Con(Q,\cdot)$ *the following statement holds*

$$(\forall x \in Q)(\forall y \in Q)(x\theta y \Leftrightarrow x \cdot y^{-1} \in F(\theta)).$$

□

For $n \geq 3$ the following proposition holds:

4.4. Theorem: *Let $n \geq 3$, let (Q, A) be an n-group and $(Q, \{\cdot, \varphi, b\})$ its* **arbitrary** *associated nHG-algebra. Further on, let*

$$\hat{L} \overset{def}{=} \{H | (H, \cdot) \lhd (Q, \cdot) \wedge \varphi(H) = H\} \text{ [:(2) from 2.1].}$$

In addition, let $^{-1}$ be the inversing operation in the group $(Q, \cdot)$. Then there is **exactly one bijection** *$\hat{F}$ of the set $Con(Q, A)$ onto the set $\hat{L}$ such that for every $\theta \in Con(Q, A)$ the following statement holds*

$$(\forall x \in Q)(\forall y \in Q)(x\theta y \Leftrightarrow x \cdot y^{-1} \in \hat{F}(\theta)).$$

Proof. By Theorem 4.1, using Proposition 4.3 and Proposition 3.1, we conclude that for every $\theta \in Con(Q, \cdot)$ the following equivalence holds

$$\theta \in Con(Q, A) \Leftrightarrow \theta \in Con(Q, \cdot) \wedge \varphi(F(\theta)) = F(\theta),$$

where F is **uniquely determined bijection** of the set $Con(Q, \cdot)$ onto the set L such that for every $\theta \in Con(Q, \cdot)$ the following statement holds

$$(\forall x \in Q)(\forall y \in Q)(x\theta y \Leftrightarrow x \cdot y^{-1} \in F(\theta))$$

[:Proposition 4.3]. Whence, by the following conventions

$$F(\theta) \in FCon(Q, A) \overset{def}{\Leftrightarrow} \theta \in Con(Q, A) \text{ and}$$

$$F(\theta) \in FCon(Q, \cdot) \overset{def}{\Leftrightarrow} \theta \in Con(Q, \cdot),$$

we conclude that for every $\theta \in Con(Q, \cdot)$ the following equivalence holds

$$F(\theta) \in FCon(Q, A) \Leftrightarrow F(\theta) \in FCon(Q, \cdot) \wedge \varphi F(\theta) = F(\theta),$$

whence, by the definition of the set $\hat{L}$ [and the set L], we conclude that for every $\theta \in Con(Q, \cdot)$ the following equivalence holds

$$F(\theta) \in FCon(Q, A) \Leftrightarrow F(\theta) \in \hat{L}$$

[:$FCon(Q, \cdot) = L$], i.e., that the following equality holds

$$FCon(Q, A) = \hat{L}.$$

Restriction $\hat{F}$ of the bijection $F : Con(Q, \cdot) \to L$ defined by

$$\hat{F}(\theta) = F(\theta) \text{ for every } \theta \in Con(Q, A),$$

is thus a **uniquely determined bijection** of the set $Con(Q, A)$ onto the set $\hat{L}$ such that for every $\theta \in Con(Q, A)$ the following statement holds

$$(\forall x \in Q)(\forall y \in Q)(x\theta y \Leftrightarrow x \cdot y^{-1} \in \hat{F}(\theta)).$$

5. On the lattice of congruences of an n-group

By Proposition 2.1, Proposition 4.3 and by Theorem 4.4, we conclude that the following proposition holds:

5.1. Theorem: *Let $n \geq 3$, let (Q, A) be an n-group and let $(Q, \{\cdot, \varphi, b\})$ be its* **arbitrary** *associated nHG-algebra. Further on, let $(L, \odot, \cap)$ and $(\hat{L}, \odot, \cap)$ be the lattices from Proposition 2.1, and*

$$F : Con(Q, \cdot) \to L \text{ and}$$
$$\hat{F} : Con(Q, A) \to \hat{L}$$

bijections described in the proof of Theorem 4.4. In addition let

$$\theta_1 \dot{\smile} \theta_2 \overset{def}{=} F^{-1}(F(\theta_1) \odot F(\theta_2)) \text{ and}$$
$$\theta_1 \underset{\cdot}{\frown} \theta_2 \overset{def}{=} F^{-1}(F(\theta_1) \cap F(\theta_2)).$$

Then: (i) $(Con(Q, \cdot), \dot{\smile}, \underset{\cdot}{\frown})$ is a modular lattice[13]*; (ii) $(Con(Q, A), \dot{\smile}, \underset{\cdot}{\frown})$ is a sublattice of the lattice $(Con(Q, \cdot), \dot{\smile}, \underset{\cdot}{\frown})$; (iii) F is an isomorphism of the lattice $(Con(Q, \cdot), \dot{\smile}, \underset{\cdot}{\frown})$ to the lattice $(L, \odot, \cap)$; and (iv) $\hat{F}$ is an isomorphism of the lattice $(Con(Q, A), \dot{\smile}, \underset{\cdot}{\frown})$ to the lattice $(\hat{L}, \odot, \cap)$.*

5.2. Remark: *For all $\theta_1, \theta_2 \in Con(Q, \cdot)$ the following equalities hold*

$$\theta_1 \dot{\smile} \theta_2 = \theta_1 \circ \theta_2 \text{ and}$$
$$\theta_1 \underset{\cdot}{\frown} \theta_2 = \theta_1 \cap \theta_2,$$

where

$$\theta_1 \circ \theta_2 \overset{def}{=} \{(x, y) | (\exists z \in Q)((x, z) \in \theta_1 \wedge (z, y) \in \theta_2)\} \text{ and}$$
$$\theta_1 \cap \theta_2 \overset{def}{=} \{(x, y) | (x, y) \in \theta_1 \wedge (x, y) \in \theta_2\}.$$

[The sketch of the proof of the proposition $(\forall \theta_1 \in Con(Q, \cdot))$ $(\forall \theta_2 \in Con(Q, \cdot))$ $\theta_1 \dot{\smile} \theta_2 = \theta_1 \circ \theta_2$: a) $(\forall x \in Q)(\exists z \in Q)$ $x \cdot z^{-1} \in F(\theta)$; b)

[13] Well known modular lattice of congruences of the group $(Q, \cdot)$.

$(\forall x \in Q)(\forall y \in Q)\ (\exists z \in Q)\ (x \cdot z^{-1} \in F(\theta_1) \wedge x \cdot y^{-1} \in F(\theta_1) \odot F(\theta_2) \Rightarrow z \cdot y^{-1} \in F(\theta_2)$; c) $x \cdot y^{-1} \in F(\theta_1) \odot F(\theta_2) \Leftrightarrow (\exists z \in Q)\ (x \cdot z^{-1} \in F(\theta_1) \wedge z \cdot y^{-1} \in F(\theta_2))$; and d) $(x,y) \in \theta_1 \smile \theta_2 \Leftrightarrow (x,y) \in F^{-1}(F(\theta_1) \odot F(\theta_2)) \Leftrightarrow x \cdot y^{-1} \in F(F^{-1}(F(\theta_1) \odot F(\theta_2))) \Leftrightarrow (\exists z \in Q)(x \cdot z^{-1} \in F(\theta_1) \wedge z \cdot y^{-1} \in F(\theta_2)) \Leftrightarrow (\exists z \in Q)((x,z) \in \theta_1 \wedge (z,y) \in \theta_2) \Leftrightarrow (x,y) \in \theta_1 \circ \theta_2$.]

5.3. Remark: *For every n-group (Q, A), $n \geq 3$, there is a group $(\bar{Q}, \cdot)$ and its normal subgroup $(H, \cdot)$ such that: 1) $Q \in \bar{Q}/H$; 2) the factor-group $(\bar{Q}/H, \square)$ [of the group $(\bar{Q}, \cdot)$ over H] is a finite cyclic group; and 3) for every $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \ldots \cdot x_n$ [: Post's coset theorem, 1940; for example [6, 7]]. In [4], Monk and Sioson described the lattice of congruences of the n-group (Q, A), $n \geq 3$, up to an isomorphism, by means of the lattice of normal subgroups of the group $(H, \cdot)$ which are at the same time subgroup of the group $(\bar{Q}, \cdot)$.*

6. A connection of congruence classes of the given congruence of an n-group with its associated nHG-algebras

6.1. Theorem: *Let $n \geq 3$ and let (Q, A) be an n-group. Further on, let θ be an arbitrary element of the set $Con(Q, A)$. Then,* **for every** *$C_t \in Q/\theta$* **there is** *an nHG-algebra $(Q, \{\cdot, \varphi, b\})$ associated to the n-group (Q, A) such that the following statements hold:*

(i) $(C_t, \cdot) \lhd (Q, \cdot)$;

(ii) (C_t, φ) *is a 1-groupoid; and*

(iii) (C_t, A) *is an n-subgroup of the n-group* (Q, A) *iff* $b \in C_t$.

Proof. 1° Let $\mathbf{e}$ be a $\{1, n\}$-neutral operation of the n-group (Q, A) [:1.2.4, 1.2.2]. Further on, let θ be an arbitrary congruence of the n-group (Q, A) [$\theta \in Con(Q, A)$] and let C_t [$t \in Q$] be an arbitrary class from the set Q/θ. Then, by Proposition 1.2.5, we conclude that there is a sequence c_1^{n-2} over Q such that

$$\mathbf{e}(c_1^{n-2}) = t. \tag{0}$$

Further, assume that the sequence c_1^{n-2} over Q satisfies (0). Then, by Proposition 1.3.3, the algebra $(Q, \{\cdot, \varphi, b\})$ defined with

$$x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y); \tag{1}$$

$$\varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}) \; [= A(t, x, c_1^{n-2})]; \text{ and} \tag{2}$$

$$b \overset{def}{=} A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}) \; [= A(\overset{n}{t})] \tag{3}$$

is an nHG-algebra associated to the n-group (Q, A).

2° $(C_t, \cdot)$ is a subgroup of the group $(Q, \cdot)$

Indeed:

By (1) from 1° and by the definition of the $\{1, n\}$-neutral operation of an n-groupoid [:1.2.2], we conclude that $\mathbf{e}(c_1^{n-2})$ is the neutral element of the group $(Q, \cdot)$, whence, by (0) [from 1°], we conclude that the neutral element $\mathbf{e}(c_1^{n-2})$ of the group $(Q, \cdot)$ belongs to C_t, i. e., that

$$\mathbf{e}(c_1^{n-2}) \in C_y. \tag{4}$$

Further on, if f is an inversing operation in the n-group (Q, A) [:1.2.6], then the unary operation $^{-1}$ in Q, defined with

$$x^{-1} \overset{def}{=} f(c_1^{n-2}, x), \tag{5}$$

is an inversing operation in the group $(Q, \cdot)$. In addition, for every $\theta \in P(Q^2)$ the following implication holds

$$\theta \in Con(Q, A) \Rightarrow \theta \in Con(Q, f) \tag{6}$$

[:[12]; 1.4.4]. Finally, using the statements connected with (1) [from 1°] and connected with (4)-(6), and also by Proposition 1.2.6, we conclude that for every $x, y \in Q$ the following series of implications holds

$$\begin{aligned}
x \in C_t \wedge y \in C_t \; &\Rightarrow x \theta \mathbf{e}(c_1^{n-2}) \wedge y \theta \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow f(c_1^{n-2}, x) \theta f(c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \wedge y \theta \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow A(f(c_1^{n-2}, x), c_1^{n-2}, y) \theta A(f(c_1^{n-2}, \mathbf{e}(c_1^{n-2})), c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \\
&\Rightarrow A(f(c_1^{n-2}, x), c_1^{n-2}, y) \theta \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow x^{-1} \cdot y \theta \mathbf{e}(c_1^{n-2}) \\
&\Rightarrow x^{-1} \cdot y \in C_t,
\end{aligned}$$

whence we conclude that $(C_t,\cdot)$ is a subgroup of the group $(Q,\cdot)$.

3° $(C_t,\cdot) \lhd (Q,\cdot)$.

Indeed:

Let a be an arbitrary element from Q and let x be an arbitrary element from C_t. Then, by Proposition 1.4.4, Proposition 1.2.6, (1) [from 1°] and (5) [from 2°], we conclude that the following series of equivalences holds

$$\begin{aligned} x \in C_t &\Leftrightarrow x\theta \mathbf{e}(c_1^{n-2}) \\ &\Leftrightarrow A(a, c_1^{n-2}, x)\theta A(a, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \\ &\Leftrightarrow A(a, c_1^{n-2}, x)\theta a \\ &\Leftrightarrow A(A(a, c_1^{n-2}, x), c_1^{n-2}, f(c_1^{n-2}, a))\theta A(a, c_1^{n-2}, f(c_1^{n-2}, a)) \\ &\Leftrightarrow A(A(a, c_1^{n-2}, x), c_1^{n-2}, f(c_1^{n-2}, a))\theta \mathbf{e}(c_1^{n-2}) \\ &\Leftrightarrow a \cdot x \cdot a^{-1} \in C_t. \end{aligned}$$

4° (C_t, φ) is a 1-groupoid.

Indeed:

By Proposition 1.4.4, (2) from 1°, by the fact that $\varphi \in Aut(Q,\cdot)$, and also since $\mathbf{e}(c_1^{n-2})$ is the neutral element of the group $(Q,\cdot)$ [:1°], we conclude that for every $x \in Q$ the following series of equivalences holds

$$\begin{aligned} x \in C_t &\Leftrightarrow x\theta \mathbf{e}(c_1^{n-2}) \\ &\Leftrightarrow A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})\theta A(\mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) \\ &\Leftrightarrow \varphi(x)\theta\varphi(\mathbf{e}(c_1^{n-2})) \\ &\Leftrightarrow \varphi(x)\theta \mathbf{e}(c_1^{n-2})) \\ &\Leftrightarrow \varphi(x) \in C_t \end{aligned}$$

whence we conclude that the statement (ii) holds.

5° Since $(Q, \{\cdot, \varphi, b\})$ [defined with (1)-(3) in 1°] is an nHG-algebra associated to the n-group (Q, A) [:1.3.3], by (i) and (ii), we conclude that for every $x_1^n \in C_t$ there is $y \in C_t$ such that the following equality holds

$$A(x_1^n) = y \cdot b.$$

Whence, since $C_t \in Q/\theta$, $\theta \in Con(Q, A)$ and, by Theorem 4.1, $\theta \in Con(Q,\cdot)$, we conclude that also the statement (iii) holds.

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