NOVI SAD J. MATH.
VOL. 28, NO. 2, 1998, 119-125

THE LOCALLY CONVEX $\mathcal{A}$-SPACES AND THEIR DUAL SPACES

Wu Junde
Department of Mathematics
Daquing Petroleum Institute
Anda 151400
China

Endre Pap
Institute of Mathematics
University of Novi Sad
21000 Novi Sad
Yugoslavia

Du Xinghua
Department of Mathematics
Daquing Petroleum Institute
Anda 151400
China

Abstract

Let E be a locally convex Hausdorff space with continuous dual E' and sequentially continuous dual E^s. In this paper, we show that if E is an $\mathcal{A}$-space, then $(E, \sigma(E, E^s)), (E, \beta(E, E^s)), (E^s, \sigma(E^s, E))$ and $(E^s, \beta(E^s, E))$ are all $\mathcal{A}$-spaces. In particular, if E is a Mazur $\mathcal{A}$-space, then $(E', \sigma(E', E))$ and $(E', \beta(E', E))$ are both $\mathcal{A}$-spaces. We apply the obtained results to generalize the Adjoint Theorem on operators with the domain being a locally convex $\mathcal{A}$-space.

AMS Mathematics Subject Classification (1991): 46A03, 46A05, 46A30
Key words and phrases: $\mathcal{A}$-space, dual space, adjoint operator

1. Introduction

Let (X, τ) be a topological vector space, a sequence $\{x_n\}$ from X is said to be τ-$\mathcal{K}$ convergent if each subsequence of $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is τ convergent to an element $x \in X$ [1, §3].

A subset B of X is said to be τ-$\mathcal{K}$ bounded if for each sequence $\{x_n\}$ from B and each scalar sequence $\{t_n\}$ such that $t_n \to 0$, the sequence $\{t_n x_n\}$ is τ-$\mathcal{K}$ convergent [1, §3]. A τ-$\mathcal{K}$ bounded subset B of X must be τ-bounded but in general the converse does not hold [1, §3].

A topological vector space (X, τ) is a $\mathcal{K}$-space if each sequence which converges to 0 is τ-$\mathcal{K}$-convergent [1, §3].

A topological vector space (X, τ) is said to be an $\mathcal{A}$-space if each τ-bounded subset of X is τ-$\mathcal{K}$ bounded [3].

Let (E, τ) be a locally convex Hausdorff space with continuous dual E' and sequentially continuous dual E^s, i.e., E^s is the space of all sequentially continuous linear functionals defined on E. If E and F are a pair of vector spaces in duality, let $\sigma(E, F)(\tau(E, F), \beta(E, F))$ be the weak topology (Mackey topology, strong topology) on E from this duality.

There are a large number of important $\mathcal{A}$-spaces, many of which are not complete or $\mathcal{K}$-spaces [3, Prop. 6 and Coroll. 6–7]. It is very interesting that if (E, τ) is a locally convex $\mathcal{A}$-space and $(E, \tau(E, E'))$ is an infrabarrelled space, then (E, τ) must be sequentially complete and must also be boundedly complete [5, Th. 2]. $\mathcal{A}$-space have been shown to enjoy many important properties [6], particular with respect to the Uniform Bounded Principle and hypocontinuity for bilinear operators [2, 3]. Now, we would like to show some new important facts for locally convex $\mathcal{A}$-spaces and their dual spaces.

Our proofs need the following lemma.

Lemma 1. *[6] Let (E, F) be a dual pair. Then all topologies on E admissible with respect to (E, F) have the same $\mathcal{K}$-bounded sets. In particular, if (E, τ) is an $\mathcal{A}$-space, then E is also an $\mathcal{A}$-space for any topology on E admissible with respect to (E, E').*

2. The Spaces $(E, \sigma(E, E^s))$ and $(E^s, \sigma(E^s, E))$

At first, we study the space $(E, \sigma(E, E^s))$.

Theorem 1. *Let (E, τ) be a locally convex space, then (E, τ), $(E, \sigma(E, E'))$ and $(E, \sigma(E, E^s))$ have the same bounded sets and the same $\mathcal{K}$-bounded subsets.*

Proof. It follows from the Mackey Theorem and Lemma 1 that (E, τ) and $(E, \sigma(E, E'))$ have the same bounded sets and the same $\mathcal{K}$-bounded subsets.

Since the topology $\sigma(E, E')$ is weaker then topology $\sigma(E, E^s)$, hence, we only need to show that $\sigma(E, E')$ bounded sets are $\sigma(E, E^s)$ bounded, $\sigma(E, E')$-$\mathcal{K}$ bounded sets are $\sigma(E, E^s)$-$\mathcal{K}$ bounded.

In fact, let $B \subseteq E$ be $\sigma(E, E')$ bounded, for each sequence $\{x_n\} \subseteq B$ and each scalar sequence $\{t_n\}$ such that $t_n \to 0$ and each $f \in E^s$, then $\{t_n x_n\}$ is τ-convergent to 0, and so it follows that

$$\lim_{n \to \infty} f(t_n x_n) = 0.$$

That is, B is a $\sigma(E, E^s)$-bounded subset of E.

Note that each $\sigma(E, E')$-$\mathcal{K}$ bounded set must be τ-$\mathcal{K}$ bounded. It is easy to show that each $\sigma(E, E')$-$\mathcal{K}$ bounded set must also be $\sigma(E, E^s)$-$\mathcal{K}$ bounded set.

From Lemma 1 and Theorem 1 it follows that:

Corollary 1. *If (E, τ) is a locally convex space, then all topologies on E admissible with respect to (E, E') or (E, E^s) have the same $\mathcal{K}$ bounded sets. In particular, if (E, τ) is an $\mathcal{A}$-space, then E is also an $\mathcal{A}$-space for any topology on E admissible with respect to (E, E') or (E, E^s).*

A locally convex sace is said to be a Mazur space if $E' = E^s$ [7, §8.6].

The following example shows that Corollary 1 generalize Lemma 1.

Example 1. *If X is a normed, barrelled, and not complete space, then $(X', \sigma(X', X))$ is an $\mathcal{A}$-space [3], but it is not a Mazur space [7, Prob. 9-3-117].*

Let $E = X'$, $\tau = \sigma(X', X)$, then $\sigma(E, E')$ is actually weaker than $\sigma(E, E^s)$.

Let τ_1 and τ_2 be two locally convex topologies on E if $(E, \tau_1)^s = (E, \tau_2)^s$, then τ_1 and τ_2 is said to be sequentially compatible.

Corollary 2. *Let (E, τ_1) and (E, τ_2) be sequentially compatible if (E, τ_1) is an $\mathcal{A}$-space, then (E, τ_2) is also an $\mathcal{A}$-space. That is, the locally convex $\mathcal{A}$-space is sequentially compatible invariant property.*

For a locally convex space (E, τ), let $\tau^+ = \sup\{\tau' : \tau'$ is a locally convex topology on E with the same convergent sequences as $\tau\}$ [8]. Webb has also shown that $(E, \tau)^s = (E, \tau^+)^s = (E, \tau^+)'$. So from Corollary 2 we have

Corollary 3. *If (E, τ) is a locally convex $\mathcal{A}$-space, then (E, τ^+) is also an $\mathcal{A}$-space.*

Next, we study the space $(E^s, \sigma(E^s, E))$.

Theorem 2. *If (E, τ) is a locally convex Hausdorff space, then $(E, \sigma(E, E^s))$ is a Mazur space.*

Proof. Let $f \in (E, \sigma(E, E^s))^s$ and $\{x_n\}$ be τ-convergent to 0, then for each $g \in E^s$, $g(x_n) \to 0$. Thus, $x_n \to 0$ in $(E, \sigma(E, E^s))$ and hence $f(x_n) \to 0$. It follows that $f \in E^s$, that is $(E, \sigma(E, E^s))^s \subseteq E^s$. Note that $(E, \sigma(E, E^s))' = E^s$ [7, §8.2], $(E, \sigma(E, E^s))^s = (E, \sigma(E, E^s))'$, $(E, \sigma(E, E^s))$ is a Mazur space.

Corollary 4. *Let (E, τ) be a locally convex Hausdorff space, then $(E^s, \beta(E^s, E))$ is sequentially complete and, hence, is an $\mathcal{A}$-space.*

Proof. Since $(E, \sigma(E, E^s))$ is a Mazur space it follows from [7, §8.6] that $(E^s, \beta(E^s, E))$ is sequentially complete and, hence, is an $\mathcal{A}$-space.

Example 2. *Let c_{oo} be the the space of all sequences which are eventually 0 and τ be the sup-norm topology. Then (c_{oo}, τ) is a normed space and, hence, is a Mazur space. We have $(c_{oo}, \tau)' = (c_{oo}, \tau)^s = l_1$. It follows from Corollary 4 that $(c_{oo}^s, \beta(c_{oo}^s, c_{oo})) = (l_1, \beta(l_1, c_{oo}))$ is an $\mathcal{A}$-space. But $(c_{oo}^s, \sigma(c_{oo}^s, c_{oo})) = (l_1, \sigma(l_1, c_{oo}))$ is not an $\mathcal{A}$-space. In fact, if e_k is the sequence with 1 in the kth coordinate and 0 elsewhere, then $\{ke_k\} \subseteq l_1$ is $\sigma(l_1, c_{oo})$ bounded, but, it is not $\sigma(l_1, c_{oo})$-$\mathcal{K}$ bounded.*

Example 2 shows that if $(E^s, \beta(E^s, E))$ is an $\mathcal{A}$-space it does not imply that $(E^s, \sigma(E^s, E))$ is also an $\mathcal{A}$-space. But, for a locally convex $\mathcal{A}$-space we have:

Theorem 3. *If (E, τ) is a locally convex $\mathcal{A}$-space, then $(E^s, \sigma(E^s, E))$ is also an $\mathcal{A}$-space.*

Proof. Let $A \subseteq E^s$ be $\sigma(E^s, E)$ bounded. For each $\{f_n\} \subseteq A$ and each scalar sequence $\{t_n\}$ such that $t_n \to 0$, pick a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $\sum_j |t_{n_j}| < \infty$. Denote $f = \sum_j t_{n_j} f_{n_j}$, then f is a linear functional defined on E. Now, we show that $f \in E^s$. In fact, if $\{x_i\}$ is τ-convergent to 0, it follows from [3, Coroll. 4] that

$$\sup_{i,n}\{|f_n(x_i)|\} = M < \infty.$$

For each $\epsilon > 0$, pick $j_0 \in N$ such that

$$M \sum_{j=j_0+1}^{\infty} |t_{n_j}| < \frac{\epsilon}{2}.$$

Note that $\{f_n\} \subseteq E^s$, there is $i_0 \in N$ such that for $i \geq i_0$ we have

$$\sum_{j=1}^{j_0} |t_{n_j} f_{n_j}(x_i)| < \frac{\epsilon}{2}.$$

Thus, for $i \geq i_0$ we have

$$|f(x_i)| = \left|\sum_j t_{n_j} f_{n_j}(x_i)\right| \leq \sum_{j=1}^{j_0} |t_{n_j} f_{n_j}(x_i)| + M \sum_{j=j_0+1}^{\infty} |t_{n_j}| < \epsilon.$$

This shows that $f \in E^s$. Therefore $(E^s, \sigma(E^s, E))$ is an $\mathcal{A}$-space.

Corollary 5. *If (E, τ) is an $\mathcal{A}$-space, then E^s is also an $\mathcal{A}$-space for any topology on E^s admissible with respect to (E^s, E).*

Corollary 6. *If (E, τ) is a Mazur $\mathcal{A}$-space, then $(E', \sigma(E', E))$ is also an $\mathcal{A}$-space.*

Corollary 7. *If (E, τ) is a Mazur $\mathcal{A}$-space, then E' is also an $\mathcal{A}$-space for any topology on E' admissible with respect to (E', E).*

3. The Adjoint Theorem

Let E, F be two locally convex Hausdorff spaces and $T : E \to F$ be a linear operator. The domains of the adjoint operator T' and sequentially adjoint operator T^s are defined to be

$$D(T') = \{y' : y' \in F', y'T \in E'\}, \quad D(T^s) = \{y' : y' \in F^s, y'T \in E^s\},$$

respectively. $T' : D(T') \to E'$ and $T^s : D(T^s) \to E^s$ are defined by $T'y' = y'T$ and $T^s y' = y'T$.

Theorem 4. *Let E and F be two locally convex Hausdorff spaces and $T : E \to F$, then $T^s : D(T^s) \to E^s$ carries $\sigma(F^s, F)$ bounded subsets of $D(T^s)$ to subsets of E^s which are uniformly bounded on $\sigma(E, E^s)$-$\mathcal{K}$ bounded subsets of E.*

Proof. Consider the spaces $(E, \sigma(E, E^s))$ and $(F, \sigma(F, F^s))$. Note that $(E, \sigma(E, E^s))' = E^s$ and $(F, \sigma(F, F^s))' = F^s$. Then it follows from [4, Th. 1] that the conclusion holds.

Theorem 4 may also been proved in an analogous way as in [4].

Corollary 8. *Let (E, τ) be an $\mathcal{A}$-space, then T^s carries $\sigma(F^s, F)$ bounded subsets of $D(T^s)$ to $\beta(E^s, E)$-$\mathcal{K}$ bounded subsets. In particular, T' carries $\sigma(F', F)$ bounded subsets of $D(T')$ to strongly bounded subsets of E'.*

References

[1] Antosik, P., Swartz, C., Matrix Methods in Analysis, Springer-Verlag, Lecture Notes in Mathematics 1113, Heidelberg, 1985.

[2] Antosik, P., Swartz, C., Boundedness and continuity for bilinear operators, Studia Sci. Math. Hungar., 29 (1994), 387–395

[3] Li Ronglu, Swartz, C., Spaces for which the uniform boundedness principle holds, Studia Sci. Math. Hungar., 27 (1992), 379–384

[4] Pap, E., The adjoint theorem on $\mathcal{A}$-spaces, Novi Sad J. Math., 26, (1996) 63–68

[5] Wu Junde, $\mathcal{K}$-bounded sets and $\mathcal{A}$-spaces, Chinese Quarter. J. Math., 12 (1998), 64–66

[6] Wu Junde, Li Ronglu, An Orlitz-Pettis theorem with applications to $\mathcal{A}$-spaces, Studia Sci. Math. Hungar., 35 (1999), (to appear)

[7] Wilansky, A., Modern Methods in Topological Vector Spaces, McGraw-Hill, New York, 1978.

[8] Webb, J.H., Sequential convergence in locally convex spaces, Proc. Camb. Phil. Soc., 64 (1968), 341–364

Received by the editors December 9, 1998.