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A NEW ALGORITHM FOR THE THREE COUNTERFEIT COINS PROBLEM

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Abstract

We consider the problem of determining the minimum number of weighings which suffice to find the counterfeit (heavier) coins in a set of n coins given a balance scale and the information that there are exactly three heavier coins present. A sequential algorithm is constructed for which the maximum number of steps differs by at most one from the information-theoretical lower bound.

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1. Introduction

Let X be a set of n coins indistinguishable except that exactly m of them are slightly heavier than the rest. All heavier (counterfeit) coins are supposed to be of equal weight and so are all good coins. We denote the set of counterfeit coins by SC. Given a balance scale, we want to find an optimal weighing algorithm, i.e. an algorithm which minimizes the maximum number of steps (weighings) which are required to determine SC. We assume that the difference in weight between a normal and a counterfeit coin is so small that we can gain no information by balancing two subsets of different cardinalities, i.e. the larger of two numerically unequal subsets is always the heavier.

The step (A, B) will mean balancing A against B, where A and B are two disjoint subsets of X of the same cardinality. There are three possible outcomes of the step:

1) $A \approx B$ which means A and B are of the same weight, i.e. they contain the same number of counterfeit coins.

2) $A > B$ which means A is heavier than B, i.e. A contains more defective coins than B.

3) $A < B$ which means B is heavier than A.

Let $c_m(n)$ be the minimum number of weighings required to find all counterfeit coins. There is a simple lower bound for $c_m(n)$ usually referred to as the information-theoretical lower bound

$$c_m(n) \geq \left\lceil \log_3 \binom{n}{m} \right\rceil . \tag{1}$$

The case $m = 1$ is a simple one. It is well known puzzle [1, 4] that

$$c_1(n) = \lceil \log_3 n \rceil . \tag{2}$$

The case $m = 2$ turned out to be much more complicated. R.Tošić [6] proved the following:

$$c_2(n) \leq \left\lceil \log_3 \binom{n}{2} \right\rceil + 1. \tag{3}$$

It is still an open question whether the information-theoretical bound is achievable for every n.

The following statements proved in [2] will be of use in the sequel:

$$n \leq 4 \cdot 3^k \Rightarrow c_2(n) \leq 2k + 2 \tag{4}$$

$$n \leq 20 \cdot 3^k \Rightarrow c_2(n) \leq 2k + 5 \tag{5}$$

L.Pyber [5] investigated the general case. He proposed an algorithm for the situation when m is an upper bound for the number of counterfeit coins and proved the following inequality:

$$c_m(n) \le \left\lceil \log_3 \binom{n}{m} \right\rceil + 15m.$$

2. The Results

The case $m = 3$ has been studied in [3, 7, 8]. The following result obtained in [3] is analogous to inequality (3) for the case $m = 2$:

$$c_3(n) \le \left\lceil \log_3 \binom{n}{3} \right\rceil + 1.$$

The proof is based on the following statements:

$$n \le 3^k + 1 \Rightarrow c_3(n) \le 3k - 1. \tag{6}$$

$$n \le 2 \cdot 3^k \Rightarrow c_3(n) \le 3k + 1. \tag{7}$$

$$n \le 4 \cdot 3^k \Rightarrow c_3(n) \le 3k + 3. \tag{8}$$

Suppose now that $A_1, A_2, \ldots, A_k$ are pairwise disjoint subsets of X such that $|A_i| = n_i$, $i = 1, 2, \ldots, k$, and we have the information that each of the subsets A_i contains exactly m_i counterfeit coins. By $c_{m_1,m_2,\ldots,m_k}(n_1, n_2, \ldots, n_k)$ we denote the minimum number of weighings required to find all counterfeit coins in this case. In the sequel, we will need this result from [3]:

$$c_{2,1}(3^k, 2 \cdot 3^{k-1}) = 3k - 1. \tag{9}$$

Our aim in this section is to improve the results (6),(7) and (8). First, we will prove some auxiliary statements.

Lemma 1.

$$c_{1,1,1}(2, 2, 2) = 2 \tag{10}$$

$$c_{1,1,1}(3, 3, 3) = 3 \tag{11}$$

$$c_{1,1,1}(4, 4, 4) = 4 \tag{12}$$

Proof.
Let A, B, C be subsets of X such that each of them contains exactly one counterfeit coin.
Proof of (10):
Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$. The first step is (a_1b_1, a_2c_1) (which means $(\{a_1, b_1\}, \{a_2, c_1\})$.

If $a_1b_1 \approx a_2c_1$, we weigh b_1 against b_2.

If $a_1b_1 \not\approx a_2c_1$, we weigh b_1 against c_2.
Regardless of the outcome of the second weighing, the counterfeit coins are identified.
Proof of (11):
This is an immediate consequence of (2).
Proof of (12):
Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$, $C = \{c_1, c_2, c_3, c_4\}$.
The first step is $(a_1a_2b_1b_2, a_3a_4c_1c_2)$.

If $a_1a_2b_1b_2 \approx a_3a_4c_1c_2$, the second step is (b_1b_2, b_3b_4). After this weighing we can apply (i).

If $a_1a_2b_1b_2 \not\approx a_3a_4c_1c_2$, the second step is (b_1b_2, c_3c_4). Then we can apply (i) again. □

Lemma 2. *Let A, B be subsets of X and we have the information that each of the sets A, B contains at least one counterfeit coin (total number of counterfeit coins in $A \cup B$ is three).*
(i) If $|A| \leq 2 \cdot 3^k$, $|B| \leq 2 \cdot 3^k$, we can find counterfeit coins using $3k + 2$ weighings.
(ii) If $|A| \leq 3^{k+1}$, $|B| \leq 3^{k+1}$, we can find counterfeit coins using $3k + 3$ weighings.
(iii) If $|A| \leq 4 \cdot 3^k$, $|B| \leq 4 \cdot 3^k$, we can find counterfeit coins using $3k + 4$ weighings.

Proof.
(i) The proof is by induction. It is easy to check that the statement is true for $k = 0$. Let $k > 0$ and suppose that the statement is true for all $l < k$. Let A_i, B_i, $i = 1, 2, 3$, be the subsets of A and B respectively, such that

$$|A_1| = |A_2| = |B_1| = |B_2| = 2 \cdot 3^{k-1}, \quad A_3 = A \setminus (A_1 \cup A_2), B_3 = B \setminus (B_1 \cup B_2).$$

The first two weighings are (A_1, A_2) and (B_1, B_2).

(a) If $A_1 \approx A_2$ and $B_1 \approx B_2$, the third step is (A_1, B_1).

(a.1) If $A_1 \approx B_1$, all counterfeit coins are in A_3 and B_3 and we can apply the induction hypothesis.

(a.2) If $A_1 > B_1$, then $|SC \cap A_1| = |SC \cap A_2| = |SC \cap B_3| = 1$ and according to (10), we can determine SC using $3k - 1$ additional weighings (note that $c_{1,1,1}(2,2,2) = 2 \Rightarrow c_{1,1,1}(2 \cdot 3^p, 2 \cdot 3^q, 2 \cdot 3^r) = 2 + p + q + r$).

(a.3) The case $A_1 < B_1$ is analogous to (a.2).

(b) If $A_1 \approx A_2$ and $B_1 > B_2$, the third step is (A_1, B_3) (if $|A_1| > |B_3|$ we add some good coins from B_2).

(b.1) If $A_1 \approx B_3$, all counterfeit coins are in A_3 and B_1 and we can apply the induction hypothesis.

(b.2) The case $A_1 \not\approx B_3$, is analogous to (a.2).

(c) If $A_1 > A_2$ and $B_1 > B_2$, the third step is (A_3, B_3) (we again use some good coins from B_2, if necessary).

(c.1) If $A_3 \approx B_3$, all counterfeit coins are in A_1 and B_1 and we can apply the induction hypothesis.

(c.2) The case $A_3 \not\approx B_3$, is analogous to (a.2).

(ii) The proof is quite similar to the proof of (i). We use subsets A_i and B_i such that $|A_1| = |A_2| = |B_1| = |B_2| = 3^k$, and we apply (11) instead of (10).

(iii) The proof is quite similar to the proof of (i). We use subsets A_i and B_i such that $|A_1| = |A_2| = |B_1| = |B_2| = 4 \cdot 3^{k-1}$, and we apply (12) instead of (10). □

Now we are able to prove the main theorem of the paper.

Theorem 1.

(13) $$n \le 10 \cdot 3^k \Rightarrow c_3(n) \le 3k + 5$$

(14) $$n \le 15 \cdot 3^k \Rightarrow c_3(n) \le 3k + 6$$

(15) $$n \le 20 \cdot 3^k \Rightarrow c_3(n) \le 3k + 7.$$

Proof.

Proof of (13):

Let A, B, C, D, E be the subsets of X such that $|A| = |B| = |C| = |D| =$

$\lceil n/5 \rceil$ and $E = X \setminus (A \cup B \cup C \cup D)$.
The first two steps are (A, B) and (C, D).

(a) If $A \approx B$ and $C \approx D$, the third step is (A, C).

(a.1) If $A \approx C$, then $|SC \cap E| = 3$ and according to (7), we can find counterfeit coins using additional $3k + 1$ weighings.

(a.2) If $A > C$, then $|SC \cap A| = |SC \cap B| = |SC \cap E| = 1$. According to (10), we need additional $3k + 2$ steps to find all counterfeit coins.

(a.3) The case $A < C$ is quite similar to (a.2).

(b) If $A > B$ and $C > D$, the third step is (A, C).

(b.1) The case $A \approx C$ is quite similar to (a.2).

(b.2) If $A > C$, then $|SC \cap A| = 2$ and $|SC \cap C| = 1$ and according to (5) and (2), we can determine SC using at most $3k + 2$ additional steps.

(b.3) The case $A < C$ is analogous to (b.2).

(c) If $A > B$ and $C \approx D$, the third step is (D, E'), where $E' = E \cup B'$, $B' \subseteq B$ and $|E'| = |D|$.

(c.1) If $D \approx E'$ then $SC \cap (C \cup D \cup E) = \phi$. The fourth step is (B_1, B_2), where $B_i \subseteq B$, $|B_i| \leq 3^k$ for $i = 1, 2$. We add good coins from C, if necessary.

(c.1.1) If $B_1 \approx B_2$, then $|SC \cap A| = 3$ and according to (7), we can determine SC using $3k + 1$ additional steps.

(c.1.2) If $B_1 \not\approx B_2$, then $|SC \cap A| = 2$ and according to (5) and (2) we can identify counterfeit coins using at most $3k + 1$ additional steps.

(c.2) If $D > E'$, then $|SC \cap A| = |SC \cap C| = |SC \cap D| = 1$ and according to (10) we can determine SC using $3k + 2$ additional weighings.

(c.3) If $D < E'$, each of the sets A and E' contains at least one counterfeit coin and we can apply Lemma 2 (i).
Proof of (14):
The proof is quite similar to the proof of (i).

In (a.1) we apply (6) instead of (7).

In (a.2) we apply (11) instead of (10).

In (b.2) we apply (4) instead of (5).

In (c.1.1) we apply (6) instead of (7).

In (c.1.2) we apply (9) instead of (5) and (2).

in (c.2) we apply (11) instead of (10).

In (c.3) we apply Lemma 2 (ii) instead of Lemma 2 (i).

Proof of (15):

The proof is quite similar to the proof of (i).

In (a.1) we apply (8) instead of (7).

In (a.2) we apply (12) instead of (10).

In (b.2) we apply (4) instead of (5).

In (c.1.1) we apply (8) instead of (7).

In (c.1.2) we apply (4) instead of (5).

In (c.2) we apply (12) instead of (10).

In (c.3) we apply Lemma 2 (iii) instead of Lemma 2 (i). □

A consequence of Theorem 1 is that we can now extend the set of integers for which we are able to construct an optimal algorithm.

Theorem 2. *Let n belongs to the set*

$$\bigcup_{k\geq 0}([\lceil 3^{k+1}\sqrt[3]{18}+2\rceil, 10\cdot 3^k]\cup[\lceil 3^{k+2}\sqrt[3]{2}+2\rceil, 15\cdot 3^k]\cup[\lceil 3^{k+2}\sqrt[3]{6}+2\rceil, 20\cdot 3^k]).$$

Then $c_3(n) = \left\lceil \log_3 \binom{n}{3} \right\rceil$.

Proof. Follows from Theorem 1 and from the inequalities:

$$\binom{\lceil 3^{k+1}\sqrt[3]{18}+2\rceil}{3} > 3^{3k+4}, \quad \binom{\lceil 3^{k+2}\sqrt[3]{2}+2\rceil}{3} > 3^{3k+5},$$

$$\binom{\lceil 3^{k+2}\sqrt[3]{6}+2\rceil}{3} > 3^{3k+6},$$

which are easy to prove. □

For example the information-theoretical lower bound is achieved for all n's from the intervals [9, 10], [13, 15], [18, 20], [25, 30], [36, 45], [51, 60], [72, 90], [104, 135], [149, 180] etc.

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