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# PRIMITIVE-POSITIVE MAXIMAL CLONES

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**Abstract**

It is obvious that the clone $\mathrm{Pol}\{f^\square\}$ is both a maximal clone and a primitive-positive clone if $f$ is a constant unary map, a regular permutation or $f(x, y, z) = x - y + z$ for some $p$-elementary Abelian group $(A, +, -, 0)$, $p$-prime. In this paper we show that other maximal clones are not primitive-positive clones.

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## 1. Notation and Preliminaries

By $\mathbf{N}$ we denote the set of positive integers: $\{1, 2, 3, \ldots\}$. Given $h \in \mathbf{N}$, let $\mathbf{N}_h$ denote the set $\{1, \ldots, h\}$. We shall now introduce the standard clone-theoretic notation. Let $A$ be a finite set with at least two elements.

**Operations.** $O_A^{(n)}$ is the set of all the $n$-ary operations $f : A^n \to A$ ($n \in \mathbf{N}$). $O_A := \bigcup_{n \in \mathbf{N}} O_A^{(n)}$ is the set of all the finitary operations on $A$. Given $f \in O_A$, $\mathrm{ar}(f)$ denotes the arity of $f$. If $f \in O_A^{(n)}$ satisfies the identity

$f(x_1,\dots,x_n) \approx x_i$ ($i \in \mathbf{N}_n$), we call it the $i$-th $n$-ary projection and use the notation $\pi_i^n$. $\Pi_A$ is the set of all the projections on $A$. Operation $h \in O_A^{(m)}$ is a superposition of $g \in O_A^{(n)}$ and $f_1,\dots,f_n \in O_A^{(m)}$ ($m, n \in \mathbf{N}$) iff $h(x_1,\dots,x_m) = g(f_1(x_1,\dots,x_m),\dots,f_n(x_1,\dots,x_m))$ for all $x_1,\dots,x_m \in A$. The usual notation is: $h = g(f_1,\dots,f_n)$. For $F \subseteq O_A$, let $F^{(n)}$ denote the set of all the $n$-ary operations in $F$: $F^{(n)} := \{f \in F : \mathrm{ar}(f) = n\}$.

For $f \in O_A^{(n)}$ let $f^\square = \{(x_1,\dots,x_n, f(x_1,\dots,x_n)) : x_1,\dots,x_n \in A\}$ be a $(n+1)$-ary relation referred to as the graph of $f$. For $F \subseteq O_A$, $F^\square = \{f^\square : f \in F\}$. $\Phi_A = O_A{}^\square$.

Operation $f \in O_A^{(m)}$ commutes with $g \in O_A^{(n)}$ iff

$$\begin{aligned} & f(g(x_1^{(1)},\dots,x_n^{(1)}),\dots,g(x_1^{(m)},\dots,x_n^{(m)})) \\ = \; & g(f(x_1^{(1)},\dots,x_1^{(m)}),\dots,f(x_n^{(1)},\dots,x_n^{(m)})), \end{aligned}$$

for all $x_i^{(j)} \in A$ ($i = 1\dots n$, $j = 1\dots m$). $F^*$ is the set of all $g \in O_A$ that commute with each $f \in F$.

For $a \in A$, $c_a : A \to A$ is the constant mapping $c_a(x) = a$. $\mathrm{Const}(A)$ is the set of all the constant mappings. $RP(A)$ is the set of all the fixpoint free permutations of prime order. $L(A)$ is the set of all $f \in O_A^{(3)}$ such that $f(x,y,z) = x - y + z$ for some $p$-elementary Abelian group $(A, +, -, 0)$ ($p$—prime).

**The clone of operations.** $C \subseteq O_A$ is a clone of operations on $A$ (clone for short) iff $\Pi_A \subseteq C$ and for all $g \in C^{(n)}$, $f_1,\dots,f_n \in C^{(m)}$ we have $g(f_1,\dots,f_n) \in C$.

**Relations.** $R_A^{(h)}$ denotes the set of all the nonempty $h$-ary relations on $A$: $R_A^{(h)} := \mathcal{P}(A^h) \setminus \{\emptyset\}$. If $\varrho \in R_A^{(h)}$, we set $\mathrm{ar}(\varrho) = h$. Specially, $\mathrm{ar}(\emptyset) = 0$. $R_A := \left(\bigcup_{h\in\mathbf{N}} R_A^{(h)}\right) \cup \{\emptyset\}$ is the set of all the finitary relations on $A$. For $S \subseteq R_A$ and $h \in \mathbf{N}$ put $S^{(h)} = \{\varrho \in S : \mathrm{ar}(\varrho) = h\}$.

Let $h \in \mathbf{N}$ and let $\varepsilon$ be an equivalence relation on $\mathbf{N}_h$. $h$-ary $\varepsilon$-diagonal on $A$ is the relation defined as follows: $\delta_h^\varepsilon(A) = \{\boldsymbol{x} \in A^h : (i,j) \in \varepsilon \Rightarrow \boldsymbol{x}(i) = \boldsymbol{x}(j)\}$. Let $D_A$ stand for the set of all the diagonals on $A$.

A relation $\varrho \in R_A^{(h)}$ is called totally reflexive iff $|\{a_1,\dots,a_h\}| < h$ implies

$(a_1, \ldots, a_h) \in \varrho$, for all $(a_1, \ldots, a_h) \in A^h$. A relation $\varrho \in R_A^{(h)}$ is called totally symmetric iff for all $(a_1, \ldots, a_h) \in \varrho$ and all permutations $\alpha$ of $\mathbf{N}_h$ we have $(a_{\alpha(1)}, \ldots, a_{\alpha(h)}) \in \varrho$. $\varrho \in R_A^{(h)}$ is called central iff $\varrho$ is a totally reflexive and totally symmetric relation, it is not equal to $A^h$ and has the following property: there exists a $c \in A$ such that $\{c\} \times A^{h-1} \subseteq \varrho$.

Let $h$ be an integer such that $3 \leq h \leq |A|$ and let $\Theta = \{\vartheta_1, \ldots, \vartheta_\ell\}$ ($\ell \in \mathbf{N}$) be a set of equivalence relations on $A$. We say that $\Theta$ is $h$-regular iff for all $i$, $\vartheta_i$ has exactly $h$ equivalence classes and for every $a_1, \ldots, a_\ell \in A$, $\cap_{i=1}^{\ell} a_i/\vartheta_i \neq \emptyset$ ($a/\vartheta$ is a corresponding equivalence class). $\varrho \in R_A^{(h)}$ is called an $h$-regular relation iff there is an $h$-regular set $\Theta = \{\vartheta_1, \ldots, \vartheta_\ell\}$ such that $(a_1, \ldots, a_h) \in \varrho$ iff $(\forall k \in \mathbf{N}_\ell)(\exists i, j)(i \neq j \wedge (a_i, a_j) \in \vartheta_k)$. Note that every $h$-regular relation is totally reflexive.

**Pol and Inv.** Given $\varrho \in R_A^{(h)}$ and $f \in O_A^{(n)}$ we say that $f$ preserves $\varrho$ iff for each $n$ $h$-tuples $(a_1^{(1)}, \ldots, a_h^{(1)})$, $\ldots$, $(a_1^{(n)}, \ldots, a_h^{(n)}) \in \varrho$ we have $(f(a_1^{(1)}, \ldots, a_1^{(n)}), \ldots, f(a_h^{(1)}, \ldots, a_h^{(n)})) \in \varrho$.

The polymorph of a relation $\varrho \in R_A^{(h)}$ is the set of all the operations on $A$ that preserve $\varrho$: $\mathrm{Pol}_A\{\varrho\} := \{f \in O_A : f \text{ preserves } \varrho\}$. If $S$ is a nonempty subset of $R_A$, the polymorph of $S$ is the set of all the operations on $A$ that preserve every relation in $S$: $\mathrm{Pol}_A S := \bigcap_{\varrho \in S} \mathrm{Pol}_A\{\varrho\}$. We usually abbreviate $\mathrm{Pol}_A S$ to $\mathrm{Pol}\, S$ if there is no danger of misunderstanding. For every nonempty set $S \subseteq R_A$, $\mathrm{Pol}\, S$ is a clone [1, 3].

The set of invariants of a nonempty set $F \subseteq O_A$ is the set of all the relations that are preserved by all the operations in $F$: $\mathrm{Inv}_A F := \{\varrho \in R_A : F \subseteq \mathrm{Pol}\{\varrho\}\}$. We also abbreviate $\mathrm{Inv}_A F$ to $\mathrm{Inv}\, F$ if there is no danger of misunderstanding.

The fundamental result of the theory of clones is that $C = \mathrm{Pol}\,\mathrm{Inv}\, C$ iff $C$ is a clone (see [1, 3]). The set of relations $S$ for which $S = \mathrm{Inv}\,\mathrm{Pol}\, S$ holds is often referred to as a (discrete) relational algebra. These objects are dual to clones. For $F \subseteq O_A$, put $\langle F \rangle_{\mathrm{CL}} := \mathrm{Pol}\,\mathrm{Inv}\, F$. For $S \subseteq R_A$, put $\langle S \rangle_{\mathrm{RA}} := \mathrm{Inv}\,\mathrm{Pol}\, S$. For $F \subseteq O_A$ we have $F^* = \mathrm{Pol}\, F^{\square}$ (see [3]).

**The matrix representation of a relation.** A matrix can be associated to every $\varrho \in R_A^{(h)}$, $h \in \mathbf{N}$. Let $|\varrho| = b$ and let

$$\varrho = \left\{(x_1^{(1)}, \ldots, x_h^{(1)}), \ldots, (x_1^{(b)}, \ldots, x_h^{(b)})\right\}.$$

A matrix $M_\varrho$ associated to $\varrho$ is an $h \times b$ matrix the columns of which are all the elements of $\varrho$. The rows of $M_\varrho$ are also called the rows of the relation $\varrho$. Although $M_\varrho$ is not unique, the concept is rather useful.

$$M_\varrho = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \ldots & x_1^{(b)} \\ x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(b)} \\ \vdots & \vdots & \ldots & \vdots \\ x_h^{(1)} & x_h^{(2)} & \ldots & x_h^{(b)} \end{bmatrix}$$

**Primitive-positive clones.** Let us recall that a primitive-positive formula (p. p.-formula, for short) is a formula of the form $\exists\wedge$-atomic. Let $F \subseteq O_A$ be a set of operations and let $\mathcal{F}$ be an algebraic type such that algebra $\mathbf{A} = (A, F)$ is an $\mathcal{F}$-algebra. Operation $f \in O_A$ is p. p.-definable in $\mathbf{A}$ if there is a p. p. $\mathcal{F}$-formula $\varphi(x_1, \ldots, x_n, y)$ such that $f(a_1, \ldots, a_n) = b$ iff $\mathbf{A} \models \varphi[a_1, \ldots, a_n, b]$, for all $a_1, \ldots, a_n, b \in A$. Let $\langle F \rangle_{\mathrm{PP}} = \{f \in O_A :$ $f$ is p. p.-definable in $\mathbf{A}\}$. $\langle F \rangle_{\mathrm{PP}}$ is a clone for all $F \subseteq O_A$. A clone $C$ is a p. p.-clone iff $C = \langle C \rangle_{\mathrm{PP}}$. There is a nice characterization of p. p.-clones: a clone $C$ is a p. p.-clone iff $C^\square = \langle C^\square \rangle_{\mathrm{RA}} \cap \Phi_A$ iff $C = F^*$ for some $F \subseteq O_A$ (see [1]).

**Maximal and minimal clones.** The set of all clones on $A$ forms an algebraic lattice with respect to set-theoretic operations. The lattice is both atomic and dual atomic. For historical reasons, atoms of the lattice are usually referred to as minimal clones, while dual atoms are usually referred to as maximal clones. This paper heavily depends upon the Rosenberg characterization of maximal clones. We shall introduce some special sets of relations:

$R_1$ the set of all bounded partial orders on $A$,

$R_2$ the set of selfdual relations, i.e. relations of the form $f^\square$ for some $f \in RP(A)$,

$R_3$ the set of affine relations, i.e. relations of the form $f^\square$ for some $f \in L(A)$,

$R_4$ the set of all nontrivial equivalence relations on $A$,

$R_5$ the set of all central relations on $A$, and

$R_6$ the set of all $h$-regular relations on $A$ $(h \geq 3)$.

**Theorem 1.1.** [2] *A clone $M$ is maximal iff there is a $\varrho \in R_1 \cup \ldots \cup R_6$ such that $M = \mathrm{Pol}\{\varrho\}$.* $\square$

Relations $\varrho \in R_1 \cup \ldots \cup R_6$ are often called Rosenberg relations. For all $f \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$ the clone $\langle f \rangle_{\mathrm{CL}}$ is a minimal clone, as well as a p. p.-clone.

## 2. The Result

It is obvious that the for all $f \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$ the clone $\mathrm{Pol}\{f^\square\}$ is both a maximal clone and a p. p.-clone. The major task is to show that other maximal clones are not p. p.-clones.

Suppose $C = \mathrm{Pol}\{\varrho\}$ is a maximal clone and a p. p.-clone. Then $\mathrm{Pol}\{\varrho\}$ contains nontrivial functional relations. Pick a Rosenberg relation $\varrho$ that does not belong to the classes mentioned above. The idea is to show that the $\langle\varrho\rangle_{\mathrm{RA}}$ has no nontrivial functional relations, or, equivalently, that $\langle\varrho\rangle_{\mathrm{RA}} \cap \Phi_A \subseteq D_A$.

**Lemma 2.1.** *Let $\preceq$ be a bounded partial order on $A$. Denote the least element of $\preceq$ by $0$ and the greatest element of $\preceq$ by $1$. Let $\sigma \in \langle\preceq\rangle_{\mathrm{RA}}$, let $\ell = \mathrm{ar}(\sigma)$ and let $s_1, \ldots, s_\ell$ be rows of $\sigma$. Then $S := \{(a_1, \ldots, a_\ell) \in \{0,1\}^\ell : (\forall i,j \in \mathbf{N}_\ell)(s_i \preceq s_j \Rightarrow a_i \preceq a_j)\} \subseteq \sigma$.*

*Proof.* Pick any $(a_1, \ldots, a_\ell) \in S$ and let $b = |\sigma|$.

*Case 1°* $(\forall i \in \mathbf{N}_\ell) a_i = 0$: Choose $f \in O_A^{(b)}$ such that $f(x) = 0$ for all $x \in A^b$. Obviously $f \in \mathrm{Pol}\{\preceq\} \subseteq \mathrm{Pol}\{\sigma\}$ and, thus, $(0, \ldots, 0) = (a_1, \ldots, a_\ell) \in \sigma$.

*Case 2°* $(\forall i \in \mathbf{N}_\ell)a_i = 1$: Choose $f \in O_A^{(b)}$ such that $f(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in A^b$.

*Case 3°* Let $P_0 = \{i \in \mathbf{N}_\ell : a_i = 0\} \neq \emptyset$ and $P_1 = \{i \in \mathbf{N}_\ell : a_i = 1\} \neq \emptyset$. Choose $f \in O_A^{(b)}$ such that

$$f(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \preceq \boldsymbol{s}_i \text{ for some } i \in P_0 \\ 1, & \text{otherwise.} \end{cases}$$

We have that $f \in \mathrm{Pol}\{\preceq\} \subseteq \mathrm{Pol}\{\sigma\}$ and that $f(\boldsymbol{s}_i) = a_i$ for all $i \in \mathbf{N}_\ell$. Since $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_\ell$ are rows of $\sigma$ and $f \in \mathrm{Pol}\{\sigma\}$, it follows that $(f(\boldsymbol{s}_1), \ldots, f(\boldsymbol{s}_\ell)) = (a_1, \ldots, a_\ell) \in \sigma$.

This completes the proof. □

The following lemma can be proved similarly.

**Lemma 2.2.** *Let $\varrho$ be a relation on $A$ and let $k = \mathrm{ar}(\varrho)$. Suppose there are $p, q \in A$ such that $p \neq q$ and $\{p, q\}^k \subseteq \varrho$. Let $\sigma \in \langle\varrho\rangle_{\mathrm{RA}}$, let $\ell = \mathrm{ar}(\sigma)$ and let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_\ell$ be rows of $\sigma$. Then $S := \{(a_1, \ldots, a_\ell) \in \{p, q\}^\ell : (\forall i, j \in \mathbf{N}_\ell)(\boldsymbol{s}_i = \boldsymbol{s}_j \Rightarrow a_i = a_j)\} \subseteq \sigma$.* □

Here comes another pair of twin lemmas.

**Lemma 2.3.** *Let $\preceq$ be a bounded partial order on $A$. Denote the least element of $\preceq$ by 0 and the greatest element of $\preceq$ by 1. Let $f \in O_A^{(n)}$, $n \in \mathbf{N}$, and let $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n+1}$ be rows of $f^\square$. Suppose $S := \{(a_1, \ldots, a_{n+1}) \in \{0,1\}^{n+1} : (\forall i, j \in \mathbf{N}_{n+1})(\boldsymbol{s}_i \preceq \boldsymbol{s}_j \Rightarrow a_i \preceq a_j)\} \subseteq f^\square$. Then $f \in \Pi_A$.*

*Proof.* For a start, we shall show that $(\exists i \in \mathbf{N}_n)\boldsymbol{s}_i \preceq \boldsymbol{s}_{n+1}$. Suppose, on the contrary, that $(\forall i \in \mathbf{N}_n)\neg(\boldsymbol{s}_i \preceq \boldsymbol{s}_{n+1})$. Then $(\underbrace{1, \ldots, 1}_{n}, 0), (\underbrace{1, \ldots, 1}_{n}, 1) \in S \subseteq f^\square$. Contradiction.

Dually, we can show that $(\exists j \in \mathbf{N}_n)\boldsymbol{s}_{n+1} \preceq \boldsymbol{s}_j$. Therefore, $\boldsymbol{s}_i \preceq \boldsymbol{s}_{n+1} \preceq \boldsymbol{s}_j$ for some $i, j \in \mathbf{N}_n$. This is possible iff $\boldsymbol{s}_i = \boldsymbol{s}_j$ iff $i = j$. So, $\boldsymbol{s}_i = \boldsymbol{s}_{n+1}$, i.e. $f = \pi_i^n \in \Pi_A$. □

**Lemma 2.4.** *Let $f \in O_A^{(n)}$, $n \in \mathbf{N}$, and let $s_1, \ldots, s_{n+1}$ be rows of $f^\square$. Suppose there are $p, q \in A$ such that $p \neq q$ and $S := \{(a_1, \ldots, a_{n+1}) \in \{p,q\}^{n+1} : (\forall i, j \in \mathbf{N}_{n+1})(s_i = s_j \Rightarrow a_i = a_j)\} \subseteq f^\square$. Then $f \in \Pi_A$.* □

We are now ready to prove the main statement.

**Theorem 2.1.** *Let $\varrho$ be a Rosenberg relation.* $\mathrm{Pol}\{\varrho\}$ *is a p. p.-clone iff $\varrho = g^\square$ for some $g \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$.*

*Proof.* $\Leftarrow$: Obvious.

$\Rightarrow$: Suppose $\varrho$ is a Rosenberg relation such that $\varrho \neq g^\square$ for all $g \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$. It suffices to show that $\langle \varrho \rangle_{\mathrm{RA}} \cap \Phi_A \subseteq D_A$. Pick any $f^\square \in \langle \varrho \rangle_{\mathrm{RA}} \cap \Phi_A$.

*Case 1°* $\varrho$ is a nontrivial equivalence relation: Choose $p, q \in A$ such that $p \neq q$ and $(p, q) \in \varrho$. Lemmas 2.2 and 2.4 imply $f^\square \in {\Pi_A}^\square \subseteq D_A$.

*Case 2°* $\varrho$ is an $h$-regular relation, $h \geq 3$: Choose $p, q \in A$ such that $p \neq q$ and apply lemmas 2.2 and 2.4.

*Case 3°* $\varrho$ is a central relation and $\mathrm{ar}(\varrho) \geq 2$: Let $c$ be a central element of $\varrho$, choose $a \in A$ such that $a \neq c$ and apply lemmas 2.2 and 2.4.

*Case 4°* $\varrho$ is a unary central relation and $|\varrho| \geq 2$: Choose $p, q \in \varrho$ such that $p \neq q$ and apply lemmas 2.2 and 2.4.

*Case 5°* $\varrho$ is a bounded partial order: apply lemmas 2.1 and 2.3.

This completes the proof. □

**Corollary 2.1.** *Let $C$ be a clone such that $C^*$ is a maximal clone. Then $C$ is a minimal clone. Moreover, there is a $g \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$ such that $C = \langle g \rangle_{\mathrm{CL}}$.*

*Proof.* Suppose $C^*$ is a maximal clone. Then there is a $g \in \mathrm{Const}(A) \cup RP(A) \cup L(A)$ such that $C^* = \mathrm{Pol}\{g^\square\}$. Therefore, $C^\square \subseteq \langle g^\square \rangle_{\mathrm{RA}} \cap \Phi_A$. Set $D = \langle g \rangle_{\mathrm{CL}}$. $D$ is a p. p.-clone, so $D^\square = \langle D^\square \rangle_{\mathrm{RA}} \cap \Phi_A$. Thus, we have $C \subseteq D$. Since $D$ is a minimal clone and $C \neq \Pi_A$, $C = D$. □

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