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SAFE CONVERGENCE OF TANABE'S METHOD

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Abstract

The construction of initial conditions which provide a safe convergence of the considered iterative method is one of the most important problem in finding the zeros of a given function f. In this paper initial conditions for the convergence of Tanabe's method for the simultaneous determination of all simple zeros of a polynomial are given. The established convergence conditions are of practical importance since they depend only on the available data: coefficients of a polynomial and initial approximations to the zeros.

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1. Introduction

One of the most important problems in solving polynomial equations

$$P(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

is the construction of such initial conditions which provide a safe convergence of the considered numerical algorithm. There is a lot of results in the literature which treat this subject, but the presented initial conditions most frequently depend on unattainable data. In this paper we give initial conditions for the safe convergence of Tanabe's method [6] for the simultaneous approximation of all simple zeros of a polynomial. These conditions are of a practical importance because they depend on available data: the polynomial coefficients $a_0, a_1, \ldots, a_{n-1}$, their degree n, and theinitial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of P.

Let $I_n = \{1, \ldots, n\}$ be the index set and let $z_1^{(m)}, \ldots, z_n^{(m)}$ denote approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of P at the m-th iteration. Denote by

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i^{(m)} - z_j^{(m)})} \quad (i \in I_n)$$

Weierstrass' correction appearing in the classical Weierstrass' method (also known as Durand-Kerner's or Dochev's method)

$$z_i^{(m+1)} = z_i^{(m)} - W_i^{(m)} \quad (i \in I_n;\ m = 0, 1, \ldots).$$

In this paper we will consider the following third-order method for the simultaneous approximation of all simple zeros of the polynomial P

$$(1)\, z_i^{(m+1)} = z_i^{(m)} - W_i^{(m)} \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j^{(m)}}{z_i^{(m)} - z_j^{(m)}}\right) \quad (i \in I_n;\ m = 0, 1, \ldots).$$

This method has been rediscovered by various authors and it has been derived in a different ways (see, e.g. [1], [2], [4], [5], [6]). In the literature it is most frequently referred to as Tanabe's method [6]. In a recent paper [1], it has been shown that Tanabe's method can be obtained by applying Chebyshev's method to the system of nonlinear equations

$$f_k = (-1)^k \phi_k(z_1, \ldots, z_n) - a_k = 0 \quad k = 1, 2, \ldots, n,$$

where ϕ_k denotes the k-th elementary symmetric functions:

$$\phi_k = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} z_{j_1} z_{j_2} \cdots z_{j_k}.$$

In our analysis we will deal with the following two parameters: the *maximal Weierstrass' correction* $W^{(m)} = \max_{1 \leq i \leq n} |W_i^{(m)}|$ and the *minimal* *distance* between approximations $d^{(m)} = \min_{j\neq i} |z_i^{(m)} - z_j^{(m)}|$ $(m = 0, 1, \ldots)$. For simplicity, we will sometimes omit the iteration index m and denote the quantities in the next $(m+1)$-st iteration by an additional symbol $\hat{}$ ("hat").

2. Some auxiliary results

In the convergence analysis of Tanabe's method (1) the following representation of a monic polynomial P has an important role

$$P(z) = \Big[\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1\Big] \prod_{j=1}^{n} (z - z_j), \quad W_j = \frac{P(z_j)}{\prod_{k\neq j} (z_j - z_k)}, \tag{2}$$

where $z_1, \ldots, z_n$ are distinct points. This formula is obtained using Lagrange's interpolation.

Before establishing the convergence theorem for the Tanabe method (1), we give some necessary estimates using the previous notation.

Lemma 1. *If the inequality*

$$W = \max_{1\leq i\leq n} |W_i| < \frac{d}{3n} \tag{3}$$

holds, then for $i, j \in I_n$ we have

(*i*) $\displaystyle \frac{4n-1}{3n} > \Big|1 - \sum_{j\neq i} \frac{W_j}{z_i - z_j}\Big| > \frac{2n+1}{3n};$

(*ii*) $\displaystyle |\hat{z}_i - z_i| < \frac{4n-1}{3n}|W_i| < \frac{4n-1}{9n^2}d;$

(*iii*) $\displaystyle |\hat{z}_i - z_j| > \frac{9n^2 - 4n + 1}{9n^2}d;$

(*iv*) $\displaystyle |\hat{z}_i - \hat{z}_j| > \frac{9n^2 - 8n + 2}{9n^2}d;$

(*v*) $\displaystyle \Big|\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big| < \frac{(n-1)(3n^2 + n - 1)}{(2n+1)(9n^2 - 4n + 1)};$

$$(vi)\quad \prod_{j\neq i}\Big|\frac{\hat z_i - z_j}{\hat z_i - \hat z_j}\Big| < \Big(1+\frac{4}{9(n-1)}\Big)^{n-1}.$$

Proof. Of (i): By the definition of d and (3) we find

$$\Big|1-\sum_{j\neq i}\frac{W_j}{z_i-z_j}\Big| \geq 1-\sum_{j\neq i}\frac{|W_j|}{|z_i-z_j|} \geq 1-\frac{(n-1)W}{d} > 1-\frac{n-1}{3n} = \frac{2n+1}{3n}.$$

In a similar way we estimate

$$\Big|1-\sum_{j\neq i}\frac{W_j}{z_i-z_j}\Big| \leq 1+\sum_{j\neq i}\frac{|W_j|}{|z_i-z_j|} \leq 1+\frac{(n-1)W}{d} < 1+\frac{n-1}{3n} = \frac{4n-1}{3n}.$$

Of (ii): Using (i) we obtain from (1)

$$|\hat z_i - z_i| = |W_i|\Big|1-\sum_{j\neq i}\frac{W_j}{z_i-z_j}\Big| < \frac{4n-1}{3n}|W_i| < \frac{4n-1}{9n^2}d.$$

Of (iii): By (ii) one gets

$$|\hat z_i - z_j| \geq |z_i-z_j| - |\hat z_i - z_i| > d - \frac{4n-1}{9n^2}d = \frac{9n^2-4n+1}{9n^2}d.$$

Of (iv): In regard to (ii) we find

$$|\hat z_i - \hat z_j| \geq |z_i-z_j| - |\hat z_i - z_i| - |\hat z_j - z_j| > d - 2\cdot\frac{4n-1}{9n^2}d = \frac{9n^2-8n+2}{9n^2}d.$$

Of (v): Let

$$\sigma_i = \sum_{j\neq i}\frac{W_j}{z_i-z_j}.$$

Then

$$(4)\qquad |\sigma_i| \leq \frac{(n-1)W}{d} < \frac{n-1}{3n} \quad\text{and}\quad \frac{|\sigma_i|}{1-|\sigma_i|} < \frac{n-1}{2n+1}.$$

From the iterative formula (2) we obtain

$$\frac{W_i}{\hat z_i - z_i} = -\frac{1}{1-\sum_{j\neq i}\frac{W_j}{z_i-z_j}}$$

so that by (4) it follows

$$\begin{aligned}\Big|\sum_{j=1}^{n}\frac{W_j}{\hat z_i - z_j}+1\Big| &= \Big|\frac{W_i}{\hat z_i - z_i}+\sum_{j\neq i}\frac{W_j}{\hat z_i - z_j}+1\Big| \\ &= \Big|1-\frac{1}{1-\sigma_i}+\sum_{j\neq i}\frac{W_j}{\hat z_i - z_j}\Big|\end{aligned}$$

$$\begin{aligned}&= \frac{1}{|1-\sigma_i|}\Big|\sum_{j\neq i}\frac{W_j}{\hat z_i - z_j}-\sum_{j\neq i}\frac{W_j}{z_i - z_j}-\sigma_i\sum_{j\neq i}\frac{W_j}{z_i - z_j}\Big| \\ &\leq \frac{1}{|1-\sigma_i|}\Big|\sum_{j\neq i}\frac{W_j(z_i-\hat z_i)}{(\hat z_i - z_j)(z_i - z_j)}\Big|+\frac{|\sigma_i|}{1-|\sigma_i|}\Big|\sum_{j\neq i}\frac{W_j}{\hat z_i - z_j}\Big| \\ &< \frac{3n}{2n+1}|z_i-\hat z_i|\sum_{j\neq i}\frac{|W_j|}{|\hat z_i - z_j||z_i - z_j|}+\frac{n-1}{2n+1}\sum_{j\neq i}\frac{|W_j|}{|\hat z_i - z_j|}.\end{aligned}$$

Hence, by (ii), (iii) and (3) we estimate

$$\begin{aligned}\Big|\sum_{j=1}^{n}\frac{W_j}{\hat z_i - z_j}+1\Big| &< \frac{3n}{2n+1}\cdot\frac{4n-1}{9n^2}d\cdot\frac{(n-1)W}{\frac{9n^2-4n+1}{9n^2}d\cdot d} \\ &\quad+\frac{n-1}{2n+1}\cdot\frac{(n-1)W}{\frac{9n^2-4n+1}{9n^2}d} \\ &< \frac{(n-1)(4n-1)}{(2n+1)(9n^2-4n+1)}+\frac{3n(n-1)^2}{(2n+1)(9n^2-4n+1)} \\ &= \frac{(n-1)(3n^2+n-1)}{(2n+1)(9n^2-4n+1)}.\end{aligned}$$

Of (vi): By (ii), (iv) and (3) we bound

$$\begin{aligned}\prod_{j\neq i}\Big|\frac{\hat z_i - z_j}{\hat z_i - \hat z_j}\Big| &\leq \prod_{j\neq i}\Big(1+\frac{|\hat z_j - z_j|}{|\hat z_i - \hat z_j|}\Big) < \prod_{j\neq i}\Big(1+\frac{\frac{4n-1}{9n^2}d}{\frac{9n^2-8n+2}{9n^2}d}\Big) \\ &= \Big(1+\frac{4n-1}{9n^2-8n+2}\Big)^{n-1} < \Big(1+\frac{4}{9(n-1)}\Big)^{n-1} \\ &< e^{4/9}\cong 1.56.\ \Box\end{aligned}$$

Let us define $\alpha(n)=\begin{cases}1/4 & \text{for n=3,}\\ 7/20 & \text{for } n\geq 4\end{cases}$ and $\widehat{W}=\max_i|\widehat{W}_i|$. Then we have the following assertions:

Lemma 2. *If the inequality* (3) *holds, then*

(*i*) $|\widehat{W}_i| < \alpha(n)|W_i|$;

(*ii*) $\widehat{W} < \dfrac{\hat{d}}{3n}$.

Proof. Putting $z = \hat{z}_i$ in (2), where $\hat{z}_i$ is a new approximation obtained by Tanabe's method (1), we obtain

$$P(\hat{z}_i) = (\hat{z}_i - z_i)\Big[\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big] \prod_{j\neq i}(\hat{z}_i - z_j),$$

whence

$$\widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i)\Big[\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big] \prod_{j\neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}.$$

From the last formula we obtain by (ii), (v) and (vi) of Lemma 1

$$\begin{aligned} |\widehat{W}_i| &= |\hat{z}_i - z_i|\Big|\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big| \prod_{j\neq i}\Big|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\Big| \\ &< \frac{4n-1}{3n} \cdot \frac{(n-1)(3n^2+n-1)}{(2n+1)(9n^2-4n+1)}\Big(1 + \frac{4}{9(n-1)}\Big)^{n-1}|W_i| \\ &= x(n)|W_i|, \end{aligned}$$

where $x(n)$ is given by the expression standing in front of $|W_i|$. It is easy to show that the sequence $\{x(n)\}_{n=3,4,\ldots}$ is monotonically increasing with $x(3) \cong 0.216 < 1/4$ and

$$x(n) < x(\infty) = \frac{2}{9}e^{4/9} \cong 0.3465 < 0.35 = \frac{7}{20}.$$

Therefore, we obtain $x(n) < \alpha(n)$ so that $|\widehat{W}_i| < \alpha(n)|W_i|$ and the assertion (i) is proved.

From (iv) of Lemma 1 it follows $d < \dfrac{9n^2}{9n^2 - 8n + 2}\hat{d}$. In regard to this, the inequality (i) of Lemma 2 and (3), we find for each $i \in I_n$

$$|\widehat{W}_i| < \alpha(n)|W_i| < \frac{\alpha(n)d}{3n} < \frac{\alpha(n)}{3n} \cdot \frac{9n^2\hat{d}}{9n^2 - 8n + 2} < \frac{\hat{d}}{3n},$$

which proves the assertion (ii). □

3. The convergence theorem

In this section we use the assertions of Lemmas 1 and 2 to state the following convergence theorem for Tanabe's method (1):

Theorem 1. *Tanabe's method* (1) *is convergent under the condition*

$$W^{(0)} < \frac{d^{(0)}}{3n}. \tag{5}$$

Proof. From the iterative formula (1) we see that the iterative corrections $C_i^{(m)}$ are given by

$$C_i^{(m)} = W_i^{(m)}\left(1 - \sum_{j\neq i} \frac{W_j^{(m)}}{z_i^{(m)} - z_j^{(m)}}\right) \quad (i \in I_n). \tag{6}$$

Now we will show that the sequences $\{|C_i^{(m)}|\}$ $(i = 1, \ldots, n)$ are monotonically decreasing under the condition (5). Starting from (6) and omitting iteration indices we find by (ii) of Lemma 1 (which is valid because (ii) of Lemma 2 holds under the condition (5))

$$|C_i| = \left|W_i\Bigl(1 - \sum_{j\neq i} \frac{W_j}{z_i - z_j}\Bigr)\right| < \frac{4n-1}{3n}|W_i| < \frac{4}{3}|W_i|. \tag{7}$$

In Lemma 2 (assertion (ii)) the implication $W < d/3n \Rightarrow \widehat{W} < \hat{d}/3n$ has been proved. Using a similar procedure, we prove by induction that the initial condition (5) implies the inequality $W^{(m)} < d^{(m)}/3n$ for each $m = 1, 2, \ldots$. Therefore, by (i) of Lemma 2 we obtain

$$|W_i^{(m+1)}| < \alpha(n)|W_i^{(m)}| \quad (i \in I_n;\ m = 0, 1, \ldots).$$

According to this, the inequalities (i) of Lemma 1 and by (7), we obtain (omitting the iteration indices)

$$\begin{aligned}
|\widehat{C}_i| &< \frac{4}{3}|\widehat{W}_i| < \frac{4\alpha(n)}{3}|W_i| = \frac{4\alpha(n)}{3} \cdot \frac{\left|W_i\Bigl(1 - \sum_{j\neq i} \frac{W_j}{z_i - z_j}\Bigr)\right|}{\left|1 - \sum_{j\neq i} \frac{W_j}{z_i - z_j}\right|} \\
&< \frac{4n}{2n+1}\alpha(n)|C_i| < \gamma(n)|C_i|,
\end{aligned}$$

where $\gamma(n) = 2\alpha(n) < 1$. Therefore, we have proved that the inequality

$$|C_i^{(m+1)}| < \gamma(n)|C_i^{(m)}|$$

holds for each $i = 1, \ldots, n$ and $m = 0, 1, \ldots$, where

$$\gamma(n) = \begin{cases} 1/2 & \text{for n=3,} \\ 7/10 & \text{for } \; n \geq 4 \end{cases}$$

Similarly as in [3] and [4], let us define disks $D_i^{(m)} := \{z_i^{(m+1)}; |C_i^{(m)}|\}$ for $i \in I_n$ and $m = 0, 1, \ldots$. Then, for a fixed $i \in I_n$ we have

$$\begin{aligned} D_i^{(m)} &= \{z_i^{(m)} - C_i^{(m)}; |C_i^{(m)}|\} = \{z_i^{(m-1)} - C_i^{(m-1)} - C_i^{(m)}; |C_i^{(m)}|\} = \cdots \\ &= \{z_i^{(0)} - C_i^{(0)} - C_i^{(1)} - \cdots - C_i^{(m)}; |C_i^{(m)}|\} \subset \{z_i^{(0)}; r_i^{(m)}\}, \end{aligned}$$

where

$$r_i^{(m)} = |C_i^{(0)}| + \cdots + |C_i^{(m-1)}| + 2|C_i^{(m)}|.$$

Since

$$|C_i^{(k)}| < \gamma(n)^k |C_i^{(0)}| \; (k = 1, 2, \ldots; \; \gamma(n) < 1) \, ,$$

we have

$$\begin{aligned} r_i^{(m)} &< |C_i^{(0)}|(1 + \gamma(n) + \cdots + \gamma(n)^m + \gamma(n)^m) \\ &< |C_i^{(0)}|\Big(\frac{1}{1-\gamma(n)} + \gamma(n)\Big) = g_n|C_i^{(0)}|, \end{aligned}$$

where

$$g_n = g(\gamma(n)) = \frac{1 + \gamma(n) - \gamma(n)^2}{1 - \gamma(n)} = \begin{cases} 2.5 & \text{for n=3,} \\ \\ \dfrac{121}{30} & \text{for } \; n \geq 4. \end{cases}$$

Therefore, for each $i \in I_n$ we have the inclusion

$$D_i^{(m)} \subset S_i := \{z_i^{(0)}; g_n|C_i^{(0)}|\},$$

which means that the disk S_i contains all disks $D_i^{(m)}$ $(m = 0, 1, \ldots)$.

The sequence $\{z_i^{(m)}\}$ of the centers of the disks $D_i^{(m)}$ forms a Cauchy's sequence in the disk $S_i \supset D_i^{(m)}$ $(m = 0, 1, \ldots)$. Since the metric subspace S_i

is complete (as a closed set in C), there exists a unique point $z_i^* \in S_i$ such that

$$z_i^{(m)} \to z_i^* \quad \text{as} \quad m \to \infty \quad \text{and} \quad z_i^* \in S_i.$$

In the limit case Tanabe's iterative formula (1) reduces to

$$z_i^* = z_i^* - P(z_i^*)\left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_i^*}{z_i^* - z_j^*}\right) \quad (i \in I_n), \tag{8}$$

where $W_i^* = W_i(z_i^*)$. According to (i) of Lemma 1 (which holds due to (5)) there follows

$$\left|1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| \in \left(\frac{2n+1}{3n}, \frac{4n-1}{3n}\right)$$

so that from (8) we obtain $P(z_i^*) = 0$ for each $i \in I_n$. Therefore, the limit points $z_1^*, \ldots, z_n^*$ of the sequences $\{z_1^{(m)}\}, \ldots, \{z_n^{(m)}\}$ are, actually, the zeros of the polynomial P.

To prove the theorem it is necessary to show that each of the sequences $\{z_i^{(m)}\}$ $(i \in I_n)$ converges to one and only one zero of P. Since $z_i^{(m)} \in S_i$ for each $i \in I_n$ and $m = 0, 1, \ldots$, it suffices to prove that the disks $S_1, \ldots, S_n$ are mutually disjoint, that is,

$$|\text{mid } S_i - \text{mid } S_j| > \text{rad } S_i + \text{rad } S_j = g_n(|C_i^{(0)}| + |C_j^{(0)}|) \quad (i \neq j).$$

According to (7) we have

$$|C_i^{(0)}| < \frac{4}{3}|W_i^{(0)}| \leq \frac{4}{3}W^{(0)},$$

wherefrom

$$d^{(0)} > 3nW^{(0)} > \frac{9n}{4}|C_i^{(0)}|.$$

Hence, since $\dfrac{9n}{8} > g_n$ for each $n \geq 3$,

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > \frac{9n}{8}(|C_i^{(0)}| + |C_j^{(0)}|) > g_n(|C_i^{(0)}| + |C_j^{(0)}|) = \text{rad } S_i + \text{rad } S_j.$$

Therefore, the inclusion disks $S_1, \ldots, S_n$ are disjoint, which completes the proof of the theorem. □

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