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ON THE CONVERGENCE OG HALLEY-LIKE METHOD

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Abstract

The theory of point estimation treating the initial conditions for the safe convergence of Halley-like method for the simultaneous determination of polynomial zeros is considered. Applying a general approach which makes use of corrections, the convergence conditions for this method are stated. These conditions are computationally verifiable: they depend only on the coefficients of a polynomial and initial approximations to the zeros.

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1. Introduction

The construction of initial conditions which provide a safe convergence of the considered numerical algorithm is one of the most important problems in solving algebraic equations. These conditions should be stated in such a way that they depend only on attainable data, for example, on the coefficients of a given polynomial and initial approximations $\mathbf{z}^{(0)}$. Theory of point estimation, which deals with the mentioned problems, introduces *approximate zeros* as initial points which provide a safe convergence of the considered iterative methods for the simultaneous approximations of all polynomial zeros. A number of results on this subject has been presented in [2]–[6], [7], [8]–[10].

In this paper we consider the monic algebraic polynomials of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in C)$$

which have only simple zeros. Most of iterative methods for the simultaneous approximation of simple zeros of a polynomial can be expressed in the form

$$(1) \qquad z_i^{(m+1)} = z_i^{(m)} - C_i(z_1^{(m)}, \ldots, z_n^{(m)}) \quad (i \in I_n;\ m = 0, 1, \ldots),$$

where $I_n = \{1, \ldots, n\}$ is the index set and $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to simple zeros $\zeta_1, \ldots, \zeta_n$ respectively, obtained in the mth iterative step. The term

$$C_i^{(m)} = C_i(z_1^{(m)}, \ldots, z_n^{(m)})$$

is called the *iterative correction*. For simplicity, we will sometimes omit the iteration index m and denote the quantities in the latter $(m+1)$-st iteration by an additional symbol $\widehat{\ }$ ("hat").

Let us introduce the real function g by

$$g(\gamma) = \frac{1 + \gamma - \gamma^2}{1 - \gamma}, \ \gamma \in (0, 1).$$

The following convergence theorem is proved in [4]:

Theorem 1. *Let C_i be the iterative correction term of the form $C_i(z) = P(z)/F(z)$ with $F(z) \neq 0$ for $z = \zeta_i$ and $z = z_i^{(m)}$ $(i \in I_n;\ m = 0, 1, \ldots)$. If for each $i, j \in I_n$ and $m = 0, 1, \ldots$ the following inequalities*

$$(i) \ \ |C_i^{(m+1)}| < \gamma |C_i^{(m)}| \ (\gamma < 1),$$

$$(ii) \ |z_i^{(0)} - z_j^{(0)}| > g(\gamma)(|C_i^{(0)}| + |C_j^{(0)}|) \ (i \neq j),$$

are satisfied under some initial conditions, then the iterative process (1) *is convergent.*

Let us note that the class of iterative methods considered in Theorem 1 is rather wide, and most frequently includes the methods used for finding polynomial zeros, simultaneously.

The initial conditions in the case of polynomials should be a function of the polynomial coefficients $\mathbf{a} = (a_0, \ldots, a_{n-1})$, its degree n and initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of P. For $m = 0, 1, \ldots$ let

$$d^{(m)} = \min_{\substack{i,j \in I_n \\ j \neq i}} |z_i^{(m)} - z_j^{(m)}|$$

be the minimal distance between approximations obtained in the mth iteration, and let

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i^{(m)} - z_j^{(m)})}, \quad w^{(m)} = \max_{1 \leq j \leq n} |W_j^{(m)}|.$$

As in the papers [2]–[5], [7], [8], [10], we will restrict initial conditions to the form of the inequality

$$w^{(0)} < c(n) d^{(0)}, \tag{2}$$

where $c(n)$ is a quantity which depends only on the polynomial degree n. The motivation and discussion about initial conditions of the form (2) have been given in [5]. Throughout this paper we will always assume that the polynomial degree n is ≥ 3.

2. Halley-like method

Let us introduce the denotations

$$G_{k,i} = \sum_{j \neq i} \frac{W_j}{(z_i - z_j)^k} \ (k = 1, 2), \quad t_i = \frac{W_i G_{2,i}}{(1 + G_{1,i})^2}, \quad q = \frac{n-1}{4n^2}.$$

Ellis and Watson have proposed in [1] the following iterative method for the simultaneous determination of all simple zeros of a polynomial P :

$$\hat{z}_i = z_i - \frac{W_i}{(1+G_{1,i})(1+t_i)} \quad (i \in I_n). \tag{3}$$

This formula can also be derived by applying the well known Halley method to the function

$$h_i(z) = W_i + (z - z_i)\Big(1 + \sum_{j \neq i} \frac{W_j}{z - z_j}\Big) \Big(= \frac{P(z)}{\prod_{j\neq i}(z - z_j)}\Big).$$

For this reason, the iterative method (3) is referred to as *Halley-like method.* Let us note that the iterative formula (3) is a special case of a one-parameter family of simultaneous methods

$$\hat{z}_i = z_i - \frac{(\alpha+1)W_i}{(1+G_{1,i})(\alpha + \sqrt{1+2(\alpha+1)t_i}\ )} \quad (i \in I_n), \tag{4}$$

which is obtained for $\alpha = -1$ (applying a limiting operation) (see [6]). It has been proved in [6] that the order of convergence of the family of iterative method (4) is four.

3. Some necessary lemmas

In what follows we apply Theorem 1 and an initial condition of the form (2) to state the convergence theorem for the Halley-like simultaneous method (3). Before establishing the main results, we give two necessary lemmas.

Lemma 1. *Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P of the degree n, and let $\hat{z}_1, \ldots, \hat{z}_n$ be the new respective approximations obtained by the iterative method* (3). *If the inequality*

$$w = \max_{1\le i\le n} |W_i| < \frac{d}{3n} \tag{5}$$

holds, then for $i, j \in I_n$ we have

(*i*) $\quad \frac{4}{3} > |1 + G_{1,i}| > \frac{2}{3};$

$(ii)\quad |G_{2,i}| \le \dfrac{(n-1)w}{d^2};$

$(iii)\quad |t_i| < q \le \dfrac{1}{18};$

$(iv)\quad |\hat{z}_i - z_i| = |C_i| < \frac{8}{5}|W_i| < \dfrac{8d}{15n}.$

Proof. According to the definition of the minimal distance d and the inequality (5), we have

$$\sum_{j\ne i}\frac{|W_j|}{|z_i - z_j|} \le \frac{(n-1)w}{d}$$

so that we estimate

$$|1+G_{1,i}| \ge 1 - \sum_{j\ne i}\frac{|W_i|}{|z_i - z_j|} \ge 1 - \frac{(n-1)w}{d} > 1 - \frac{1}{3} = \frac{2}{3},$$

$$|1+G_{1,i}| \le 1 + \sum_{j\ne i}\frac{|W_i|}{|z_i - z_j|} \le 1 + \frac{(n-1)w}{d} < 1 + \frac{1}{3} = \frac{4}{3},$$

$$|G_{2,i}| \le \sum_{j\ne i}\frac{|W_i|}{|z_i - z_j|^2} \le \frac{(n-1)w}{d^2}.$$

Thus, the assertions (i) and (ii) of Lemma 1 are proved.

Using (i), (ii) and (5) we prove (iii):

$$|t_i| = \left|\frac{W_i G_{2,i}}{(1+G_{1,i})^2}\right| < \left(\frac{3}{2}\right)^2 \frac{(n-1)w^2}{d^2} < \frac{n-1}{4n^2} = q \le \frac{1}{18}.$$

From (3) we have

$$\begin{aligned} |\hat{z}_i - z_i| &= |C_i| = \left|\frac{W_i}{(1+G_{1,i})(1+t_i)}\right| \le \left|\frac{W_i}{1+G_{1,i}}\right| \frac{1}{1-|t_i|} \\ &< \frac{|W_i|}{\frac{2}{3}} \cdot \frac{1}{1-\frac{1}{18}} < \frac{8}{5}|W_i| < \frac{8d}{15n}, \end{aligned}$$

which proves (iv) of Lemma 1. □

Lemma 2. *Under the conditions of Lemma 1 the following inequalities are valid:*

(i) $|\widehat{W}_i| < \frac{2}{9}|W_i|$;

(ii) $\hat{w} < \dfrac{\hat{d}}{3n}$.

Proof. Using Lagrangean interpolation, as in [5] we derive the following relation:

$$(6)\quad \widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i)\Big(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\Big)\prod_{j\neq i}\Big(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\Big).$$

By applying (iv) of Lemma 1, we have

$$(7)\quad |\hat{z}_i - z_j| \;\geq\; |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{8d}{15n} = \frac{15n-8}{15n}d,$$

$$(8)\quad |\hat{z}_i - \hat{z}_j| \;\geq\; |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - \frac{16d}{15n} = \frac{15n-16}{15n}d.$$

From the last inequality, and taking into account the definition of the minimal distance we find

$$(9)\qquad \hat{d} > \frac{15n-16}{15n}d \quad \text{or} \quad d < \frac{15n}{15n-16}\hat{d}.$$

From (3) we obtain

$$\frac{W_i}{\hat{z}_i - z_i} = -1 - G_{1,i} - t_i(1 + G_{1,i}) = -1 - \sum_{j\neq i}\frac{W_j}{z_i - z_j} - \frac{W_i G_{2,i}}{1 + G_{1,i}},$$

so that

$$\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} = -(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(z_i - z_j)(\hat{z}_i - z_j)} - \frac{W_i G_{2,i}}{1 + G_{1,i}}.$$

Hence

$$(10)\quad \Big|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\Big| \leq |\hat{z}_i - z_i|\sum_{j\neq i}\frac{|W_j|}{|z_i - z_j||\hat{z}_i - z_j|} + \Big|\frac{W_i G_{2,i}}{1 + G_{1,i}}\Big|.$$

Using the above estimates (7) and (8) for $|\hat{z}_i - z_j|$ and $|\hat{z}_i - \hat{z}_j|$, and the inequalities (5) and (iv) of Lemma 1, we estimate

$$\Big|(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(z_i - z_j)(\hat{z}_i - z_j)}\Big| \;\leq\; |\hat{z}_i - z_i|\sum_{j\neq i}\frac{|W_j|}{|z_i - z_j||\hat{z}_i - z_j|}$$

$$(11)\qquad\qquad < \frac{8(n-1)}{3n(15n-8)}.$$

According to (i)–(ii) of Lemma 1 and (5) we obtain

$$\left|\frac{W_i G_{2,i}}{1+G_{1,i}}\right| < \frac{n-1}{6n^2}. \tag{12}$$

From (10) we get by (11) and (12)

$$\left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i} \frac{W_j}{\hat{z}_i - z_j}\right| < \frac{8(n-1)}{3n(15n-8)} + \frac{n-1}{6n^2} \tag{13}$$

and

$$\begin{aligned} \left|\prod_{j\neq i}\Big(1+\frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\Big)\right| &\leq \prod_{j\neq i}\Big(1+\frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\Big) < \left(1+\frac{\frac{8d}{15n}}{\frac{(15n-16)d}{15n}}\right)^{n-1} \\ &= \Big(1+\frac{8}{15n-1}\Big)^{n-1}. \end{aligned} \tag{14}$$

Using (iv) of Lemma 1, (13) and (14), from (6) we obtain

$$\begin{aligned} |\widehat{W}_i| &\leq |\hat{z}_i - z_i|\left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\right|\left|\prod_{j\neq i}\Big(1+\frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\Big)\right| \\ &< \frac{8}{5}|W_i|\Big[\frac{8(n-1)}{3n(15n-8)} + \frac{n-1}{6n^2}\Big]\Big(1+\frac{8}{15n-16}\Big)^{n-1}, \end{aligned}$$

that is,

$$|\widehat{W}_i| < f(n)|W_i|, \tag{15}$$

where

$$f(n) = \frac{8}{5}\Big[\frac{8(n-1)}{3n(15n-8)} + \frac{n-1}{6n^2}\Big]\Big(1+\frac{8}{15n-16}\Big)^{n-1}.$$

The function f is monotonically decreasing for $n \geq 3$, so that we have

$$f(n) \leq f(3) = 0.2216... < \frac{2}{9} \quad \text{for all } n \geq 3.$$

Therefore, from (15) it follows $|\widehat{W}_i| < \frac{2}{9}|W_i|$ and the assertion (i) of Lemma 2 is proved. Using this inequality and the inequalities (5) and (9) we prove the assertion (ii):

$$\hat{w} < \frac{2}{9}w < \frac{2}{9}\cdot\frac{d}{3n} < \frac{2}{9}\cdot\frac{1}{3n}\cdot\frac{15n}{15n-16}\hat{d} < \frac{\hat{d}}{3n}. \ \square$$

4. The convergence theorem

Now we give the main result concerning the convergence of the Halley-like method (3).

Theorem 2. *The Halley-like method* (3) *is convergent under the condition*

$$w^{(0)} < \frac{d^{(0)}}{3n}. \tag{16}$$

Proof. In Lemma 2 (assertion (ii)) we have proved the implication

$$w < \frac{d}{3n} \Rightarrow \hat{w} < \frac{\hat{d}}{3n}.$$

Similarly, we prove by induction that the condition (16) implies the inequality $w^{(m)} < d^{(m)}/3n$ for each $m = 1,2,\ldots$. Therefore, all assertions of Lemmas 1 and 2 hold for each $m = 1,2,\ldots$ if the initial condition (16) is valid. In particular, the following inequalities

$$|W_i^{(m+1)}| < \frac{2}{9}|W_i^{(m)}| \tag{17}$$

and

$$|C_i^{(m)}| = |z_i^{(m+1)} - z_i^{(m)}| < \frac{8}{5}|W_i^{(m)}| \tag{18}$$

hold for $i \in I_n$ and $m = 0,1,\ldots$.

From the iterative formula (3) we see that the corrections $C_i^{(m)}$ are expressed by

$$C_i^{(m)} = \frac{W_i^{(m)}}{(1+G_{1,i}^{(m)})(1+t_i^{(m)})}, \tag{19}$$

where

$$G_{k,i}^{(m)} = \sum_{j\neq i} \frac{W_i^{(m)}}{(z_i^{(m)} - z_j^{(m)})^k}, \ (k=1,2), \quad t_i^{(m)} = \frac{W_i^{(m)} G_{2,i}^{(m)}}{(1+G_{1,i}^{(m)})^2}.$$

Now we prove that the sequences $\{|C_i^{(m)}|\}$ $(i \in I_n)$ are monotonically decreasing.

Omitting the iteration index for simplicity, from (19) we find by (17) and (18)

$$\begin{aligned}|\widehat{C}_i| &< \frac{8}{5}|\widehat{W}_i| < \frac{8}{5}\cdot\frac{2}{9}|W_i| = \frac{16}{45}|W_i| \\ &= \frac{16}{45}\left|\frac{W_i}{(1+G_{1,i})(1+t_i)}\right||(1+G_{1,i})(1+t_i)|,\end{aligned}$$

so that

(20) $$|\widehat{C}_i| < \frac{16}{45}|C_i||y_i|,$$

where we put

$$y_i = (1+G_{1,i})(1+t_i).$$

Hence, by (i) and (iii) of Lemma 1,

$$|y_i| < |1+G_{1,i}|(1+|t_i|) < \frac{4}{3}\Big(1+\frac{1}{18}\Big) = \frac{38}{27}.$$

Now from (20) we get

$$|\widehat{C}_i| < \frac{16}{45}|C_i||y_i| < \frac{2}{5}\cdot\frac{38}{27}|C_i| < 0.51|C_i|.$$

Therefore, the constant γ which appears in Theorem 1 is equal to $\gamma = 0.51$. In this way we have proved the inequality

$$|C_i^{(m+1)}| < 0.51|C_i^{(m)}|,$$

which holds for each $i = 1,\dots,n$ and $m = 0,1,\dots$.

The quantity $g(\gamma)$ appearing in (ii) of Theorem 1 is equal to $g(0.51) \approx 2.55$. It remains to prove the disjunctivity of the inclusion disks

$$S_1 = \{z_1^{(0)}; 2.55|C_1^{(0)}|\},\dots,S_n = \{z_n^{(0)}; 2.55|C_n^{(0)}|\}$$

(assertion (ii) of Theorem 1). By virtue of (iv) of Lemma 1 we have $|C_i^{(0)}| < \frac{8}{5}w^{(0)}$, wherefrom

$$\begin{aligned}d^{(0)} > (3n)w^{(0)} &> \frac{5}{8}3n|C_i| \geq \frac{15n}{16}(|C_i^{(0)}| + |C_j^{(0)}|) \\ &> g(0.51)(|C_i^{(0)}| + |C_j^{(0)}|).\end{aligned}$$

This means that

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > g(0.51)(|C_i^{(0)}| + |C_j^{(0)}|) = \operatorname{rad} S_i + \operatorname{rad} S_j.$$

Therefore, the inclusion disks $S_1,\dots,S_n$ are disjoint, which completes the proof of Theorem 2. □

References

[1] Ellis, G.H., Watson, L.T., A parallel algorithm for simple roots of polynomials, Comput. Math. Appl. 2 (1984), 107–121.

[2] Petković, M.S., On initial conditions for the convergence of simultaneous root finding methods, Computing 57 (1996), 163–177.

[3] Petković, M.S., Carstensen, C., Trajković, M., Weierstrass' formula and zero-finding methods, Numer. Math. 69 (1995), 353–372.

[4] Petković, M.S., Herceg, Dj., Point estimation and safe convergence of root-finding simultaneous methods, Scientific Review 21-22 (1996), 117–130.

[5] Petković, M.S., Herceg, Dj. , Ilić, S., Point estimation and some applications to iterative methods, BIT (to appear).

[6] Petković, M.S., Ilić, S., Tričković, S., A family of simultaneous zero-finding methods, Comput. Math. Appl. (to appear).

[7] Prešić, M., A convergence theorem for a method for simultaneous determination of all zeros of a polynomial, Publication de l'Institute Mathématique 28 (1980), 159–168.

[8] Zhao, F., Wang, D., The theory of Smale's point estimation and the convergence of Durand-Kerner program (in Chinese), Math. Numer. Sinica 15 (1993), 196–206.

[9] Wang, X., Han, D., On dominating sequence method in the point estimate and Smale's theorem, Scientia Sinica Ser. A (1989), 905–913.

[10] Wang, D., Zhao, F., The theory of Smale's point estimation and its application, J. Comput. Appl. Math. 60 (1995), 253–269.