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INTERPOLATION OF FUNCTION SPACES AND THE CONVERGENCE RATE ESTIMATES FOR THE FINITE DIFFERENCE METHOD[1]

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Abstract

In this work we use the interpolation theory to prove some convergence rate estimates for finite differences schemes. We consider the Dirichlet boundary value problem for a second order linear elliptic equation with variable coefficients in the unite square. We assume that the solution of the problem and the coefficients of equation belong to the corresponding Sobolev spaces.

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1. Introduction

For a class of finite difference schemes (FDS) for elliptic boundary value problems (BVP), the estimates of the convergence rates consistent with the

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smoothness of data, are of the major interest, i.e.

$$\|u-v\|_{W_p^k(\omega)} \le C h^{s-k} \|u\|_{W_p^s(\Omega)}, \qquad s \ge k. \tag{1}$$

Here $u = u(x)$ denotes the solution of the BVP, v denotes the solution of the corresponding FDS, h is discretisation parameter, $W_p^k(\omega)$ denotes the discrete Sobolev space, and C is a positive generic constant, independent of h and u.

Standard technique for derivation of such estimates (see [8],[9],[12]) is based on the Bramble–Hilbert lemma [2]. In this paper we present an alternative technique, based on the theory of interpolation of Banach spaces. Estimate (1) for a similar problem was derived in [7], by the same technique, for $k = 1, 2$ and $k \le s \le k + 2$.

2. Interpolation of Banach Spaces H_p^s, B_{pq}^s and W_p^s

Let A_0 and A_1 be two Banach spaces, linearly and continuosly embedded in a topological linear space $\mathcal{A}$. Two such spaces are called *interpolation pair* $\{A_0, A_1\}$. Consider also spaces $A_0 \cap A_1$ and $A_0 + A_1$ with the corresponding norms (see [2],[13]).

Let us consider the so-called complex interpolation method [13]. We can define the following sets of complex numbers: $S = \{z \in C : 0 < \Re(z) < 1\}$ and $\overline{S} = \{z \in C : 0 \le \Re(z) \le 1\}$. For the given interpolation pair $\{A_0, A_1\}$ we introduce the set $\mathcal{M}(A_0, A_1)$ of continuous functions $f : \overline{S} \to A_0 + A_1$, analytic in S, which satisfy the following conditions:

(i) $\sup_{z \in \overline{S}} \|f(z)\|_{A_0+A_1} < \infty$,

(ii) $f(j + it) \in A_j$, $j = 0, 1$, $t \in R$,

(iii) the mappings $t \to f(j + it)$, $j = 0, 1$, are continuous on t, and

(iv) $\|f\|_{\mathcal{M}(A_0,A_1)} = \max\left\{\sup_{t\in R} \|f(it)\|_{A_0}, \sup_{t\in R} \|f(1+it)\|_{A_1}\right\} < \infty.$

For $0 < \theta < 1$ with $[A_0, A_1]_\theta$ we denote the set of elements $a \in A_0 + A_1$ which satisfy the conditions:

(i) there exists a function $f \in \mathcal{M}(A_0, A_1)$ such that $f(\theta) = a$, and

(ii) $\|a\|_{[A_0,A_1]_\theta} = \inf_{f \in \mathcal{M}(A_0,A_1),\ f(\theta)=a} \|f\|_{\mathcal{M}(A_0,A_1)} < \infty.$

Defined that way, the space $[A_0, A_1]_\theta$ is an interpolation space.

The following assertion holds for bilinear operators [13]:

Lemma 1. *Let $A_0 \subset A_1$, $B_0 \subset B_1$, $C_0 \subset C_1$ and let $L : A_1 \times B_1 \to C_1$ be a continuous bilinear form whose restriction on $A_0 \times B_0$ is a continuous mapping with values in C_0. Then L is a continuous mapping from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ into $[C_0, C_1]_\theta$, and*

$$||L||_{[A_0,A_1]_\theta \times [B_0,B_1]_\theta \to [C_0,C_1]_\theta} \leq ||L||^{1-\theta}_{A_0 \times B_0 \to C_0} ||L||^{\theta}_{A_1 \times B_1 \to C_1}.$$

As an example of interpolation function spaces let us consider the spaces of the Bessel potentials H_p^s, the Besov spaces B_{pq}^s, and the Sobolev spaces W_p^s (see [1], [2] and [13]). The spaces H_p^s and B_{pq}^s are spaces of distributions. For $1 < p < \infty$ the Sobolev spaces W_p^s are defined in the following manner:

$$W_p^s(R^n) = \begin{cases} H_p^s(R^n), & s = 0, 1, 2, ... \\ B_{pp}^s(R^n), & 0 < s \neq \text{integer} \end{cases} \tag{2}$$

with the norm defined as

$$||f||_{W_p^s} = \left(\sum_{k<s} |f|^p_{W_p^k} + |f|^p_{W_p^s} \right)^{1/p},$$

where

$$|f|_{W_p^r} = \begin{cases} \left(\sum\limits_{|\alpha|=r} \int\limits_{R^n} |D^\alpha f(x)|^p dx \right)^{1/p}, & r = 0, 1, 2, ... \\ \left(\sum\limits_{|\alpha|=[r]} \int\limits_{R^n} \int\limits_{R^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+p(r-[r])}} dx dy \right)^{1/p}, & 0 < r \neq \text{integer}. \end{cases}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x = (x_1, \ldots, x_n) \in R^n$, $|x| = (x_1 + \ldots + x_n)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$ and $[r]$ is the integer part of r. Obviously, $W_p^s(R^n) \subset L_p(R^n)$, $s \geq 0$.

For $-\infty < s < \infty$, $1 < p < \infty$, $\varepsilon > 0$ and $1 \leq q_0 \leq q_1 \leq \infty$ the following embeddings hold [13]:

$$B^{s+\varepsilon}_{p,\infty}(R^n) \subset B^s_{p1}(R^n) \subset B^s_{pq_0}(R^n) \subset B^s_{pq_1}(R^n) \subset B^s_{p,\infty}(R^n) \subset B^{s-\varepsilon}_{p1}(R^n),$$

$$H_p^{s+\varepsilon}(R^n) \subset H_p^s(R^n) \quad \text{and}$$

$$B^s_{p,\min\{p,2\}}(R^n) \subset H^s_p(R^n) \subset B^s_{p,\max\{p,2\}}(R^n). \tag{3}$$

For $-\infty < t \leq s < \infty$, $1 < p \leq q < \infty$, $1 \leq r \leq \infty$ and $s - n/p \geq t - n/q$ we also have

$$B^s_{pr}(R^n) \subset B^t_{qr}(R^n) \quad \text{and} \quad H^s_p(R^n) \subset H^t_q(R^n).$$

The following assertion holds [13]:

Lemma 2. *For $-\infty < s_0, s_1 < \infty$, $1 < p_0, p_1 < \infty$, $1 \leq q_0 < \infty$, $1 \leq q_1 \leq \infty$ and $0 < \theta < 1$ we have*

$$\left[H^{s_0}_{p_0}(R^n), H^{s_1}_{p_1}(R^n)\right]_\theta = H^s_p(R^n) \quad \textit{and} \tag{4}$$

$$\left[B^{s_0}_{p_0 q_0}(R^n), B^{s_1}_{p_1 q_1}(R^n)\right]_\theta = B^s_{pq}(R^n), \tag{5}$$

where

$$s = (1-\theta)s_0 + s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

From (4), (5) and (2), for s_0, $s_1 \geq 0$, follows

$$\left[W^{s_0}_p(R^n), W^{s_1}_p(R^n)\right]_\theta = W^s_p(R^n), \quad s = (1-\theta)s_0 + s_1, \tag{6}$$

if s_0, s_1 and s are all integer, or fractional numbers. For $p = 2$ from (3) follows $W^s_2(R^n) = H^s_2(R^n) = B^s_{22}(R^n)$ and (6) holds without restriction.

The previous results hold for the spaces H^s_p, B^s_{pq} and W^s_p in a bounded domain $\Omega \subset R^n$ which satisfies the cone condition. Here we assume that $s \geq 0$ for H^s_p spaces, and $s > 0$ for B^s_{pq} spaces.

3. Boundary Value Problem and its Approximation

Our initial problem will be the Dirichlet BVP for a second-order linear elliptic equation with variable coefficients in the unit square $\Omega = (0,1)^2$:

$$-\sum\nolimits_{i,j=1}^{2} D_i(a_{ij} D_j u) + au = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega. \tag{7}$$

We assume that the generalized solution of the BVP belongs to the Sobolev space $W^s_2(\Omega)$, $1 \leq s \leq 4$, with the right-hand side $f(x)$ belonging to

$W_2^{s-2}(\Omega)$. Consequently, the coefficients of equation (7) must belong to the corresponding spaces of multipliers [10]

$$a_{ij} \in M(W_2^{s-1}(\Omega)), \quad a \in M(W_2^s(\Omega) \to W_2^{s-2}(\Omega)),$$

i.e., the sufficient conditions are

$$\begin{array}{ll} a_{ij} \in W_2^{s-1}(\Omega), \quad a \in W_2^{s-2}(\Omega) & \text{for } 2 < s \leq 4, \\ a_{ij} \in W_p^{s-1+\delta}(\Omega), \quad a = a_0 + \sum_{i=1}^2 D_i a_i & \text{for } 1 < s \leq 2. \end{array}$$

where $a_0 \in L_{2+\varepsilon}(\Omega)$, $a_i \in W_p^{s-1+\delta}(\Omega)$ and $\delta > 0$, $\varepsilon > 0$, $p \geq 2/(s-1)$.

We also assume that the corresponding differential operator is strongly elliptic, i.e.

$$a_{ij} = a_{ji}, \quad \sum_{i,j=1}^2 a_{ij} y_i y_j \geq c_0 \sum_{i=1}^2 y_i^2, \quad x \in \Omega, \quad c_0 = \text{const} > 0$$

and $a(x) \geq 0$ in the sense of distributions.

Let $\overline{\omega}$ be the uniform mesh in $\overline{\Omega}$ with the step size h, $\omega = \overline{\omega} \cap \Omega$ and $\gamma = \overline{\omega} \cap \Gamma$. We define the finite differences v_{x_i} and $v_{\overline{x}_i}$ in the usual manner [11]:

$$v_{x_i} = (v^{+i} - v)/h, \qquad v_{\overline{x}_i} = (v - v^{-i})/h,$$

where $v^{\pm i}(x) = v(x \pm hr_i)$, and r_i is the unit vector on the x_i axis.

We approximate BVP with the following FDS:

$$L_h v = T_1^2 T_2^2 f \ \text{ in } \ \omega, \quad v = 0 \ \text{ on } \ \gamma, \tag{8}$$

where

$$L_h v = -\frac{1}{2} \sum_{i,j=1}^2 \left[(a_{ij} v_{\overline{x}_j})_{x_i} + (a_{ij} v_{x_j})_{\overline{x}_i} \right] + (T_1^2 T_2^2 a) u$$

and T_i is the Steklov smoothing operator on x_i, i.e.

$$T_i^+ f(x) = \int_0^1 f(x + htr_i)\, dt = T_i^- f(x + hr_i) = T_i f(x + 0.5hr_i).$$

Hence, $T_i T_j f = T_j T_i f$ and $T_i^+ D_i u = u_{x_i}$, $T_i^- D_i u = u_{\overline{x}_i}$.

The finite-difference scheme (8) is the standard symmetric FDS [11] with averaged right-hand side and lowest-order coefficient. Note that for $s \leq 3$, $a(x)$ and $f(x)$ may be non-continuous, and consequently, the FDS with non-averaged data would not be well defined.

4. Convergence of Finite Difference Scheme

Let u be the solution of BVP (1) and v – the solution of FDS (8). The error $z = u - v$ satisfies the conditions

$$L_h z = \sum_{i,j=1}^{2} \psi_{ij,\bar{x}_i} + \psi \quad \text{in } \omega, \quad z = 0 \text{ on } \gamma, \tag{9}$$

where

$$\begin{aligned} \psi_{ij} &= T_i^+ T_{3-i}^2 (a_{ij} D_j u) - \tfrac{1}{2}\left(a_{ij} u_{x_j}) + a_{ij}^{+i} u_{x_j}^{+i}\right), \quad \text{and} \\ \psi &= (T_1^2 T_2^2 a) u - T_1^2 T_2^2 (au). \end{aligned}$$

Let $(v,w)_\omega = (v,w)_{L_2(\omega)} = h^2 \sum_{x\in\omega} v(x) w(x)$ and $\|v\|_\omega^2 = (v,v)_\omega$ denote the discrete inner product and the discrete L_2-norm on ω. We also define the discrete Sobolev norms

$$\|v\|^2_{W_2^1(\omega)} = \|v\|_\omega^2 + \sum_{i=1}^{2} \|v_{x_i}\|^2_{\omega_i}, \quad \|v\|^2_{W_2^2(\omega)} = \|v\|^2_{W_2^1(\omega)} + \sum_{i=1}^{2} \|v_{x_i\bar{x}_i}\|^2_\omega + \|v_{x_1 x_2}\|^2_{\omega_{12}},$$

where ω_i and ω_{12} are the subsets of $\overline{\omega}$ where the corresponding finite differences are well defined.

The following assertion holds [5]:

Lemma 3. *FDS (8) satisfy a priori estimates*

$$\|z\|_{W_2^1(\omega)} \le C \textstyle\sum_{i,j=1}^2 \|\psi_{ij}\|_{\omega_i} + \|\psi\|_\omega, \quad \textit{and} \tag{10}$$

$$\|z\|_{W_2^2(\omega)} \le C \textstyle\sum_{i,j=1}^2 \|\psi_{ij,\bar{x}_i}\|_\omega + \|\psi\|_\omega. \tag{11}$$

Note that (see [7]), we may obtain the following estimates:

$$\sum_{i,j=1}^{2} \|\psi_{ij,\bar{x}_i}\|_\omega \le C h^{s-2} \max_{ij} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(\Omega)}, \quad 3 \le s \le 4, \tag{12}$$

$$\sum_{i,j=1}^{2} \|\psi_{ij,\bar{x}_i}\|_\omega \le C h^{s-2} \max_{ij} \|a_{ij}\|_{W_2^{s-1+\varepsilon(3-s)}(\Omega)} \|u\|_{W_2^s(\Omega)}, \quad 2 \le s \le 3, \tag{13}$$

$$\sum_{i,j=1}^{2} \|\psi_{ij}\|_{\omega_i} \le C h^{s-1} \max_{ij} \|a_{ij}\|_{W_2^{s-1+\varepsilon(3-s)}(\Omega)} \|u\|_{W_2^s(\Omega)}, \quad 2 \le s \le 3, \tag{14}$$

$$\sum_{i,j=1}^{2} \|\psi_{ij}\|_{\omega_i} \le C h^{s-2} \max_{ij} \|a_{ij}\|_{W_p^{s-1+\varepsilon}(\Omega)} \|u\|_{W_2^s(\Omega)}, \quad 1 \le s \le 2. \tag{15}$$

In such a way, the problem of deriving the convergence rate estimates for FDS (8) is now reduced to estimating the term $\|\psi\|_\omega$. Let us represent ψ in the following manner [8]: for $1 < s \le 2$ we set $\psi = \psi_0 + \psi_1 + \psi_2$, where

$$\psi_0 = (T_1^2 T_2^2 a_0)u - T_1^2 T_2^2 (a_0 u) \quad \text{and}$$
$$\psi_i = (T_1^2 T_2^2 D_i a_i)u - T_1^2 T_2^2 (u D_i a_i), \quad i = 1, 2;$$

and for $2 \le s \le 3$ we set $\psi = \psi_3 + \psi_4$, where

$$\psi_3 = (T_1^2 T_2^2 a)(u - T_1^2 T_2^2 u) \quad \text{and}$$
$$\psi_4 = (T_1^2 T_2^2 a)(T_1^2 T_2^2 u) - T_1^2 T_2^2 (au).$$

The value ψ_0 at the node $x \in \omega$ can be represented in the form (note that $u \in W_2^{1+\alpha}(\Omega)$ and $a_0 \in L_{2+\varepsilon}(\Omega)$):

$$(16) \qquad \psi_0 = \frac{1}{h^2} \iint_e \Phi(\xi_1, \xi_2) a_0(\xi_1, \xi_2)[u(x_1, x_2) - u(\xi_1, \xi_2,)] \, d\xi_1 d\xi_2$$

where $e = (x_1 - h, x_1 + h) \times (x_2 - h, x_2 + h)$ and

$$\Phi(\xi_1, \xi_2) = \left(1 - \frac{|\xi_1 - x_1|}{h}\right)\left(1 - \frac{|\xi_2 - x_2|}{h}\right).$$

Now, from (16) follows:

$$|\psi_0| \le \frac{C}{h} \|a_0\|_{L_2(e)} \|u\|_{L_\infty(e)} \le \frac{C}{h} \|a_0\|_{L_2(e)} \|u\|_{L_\infty(\Omega)}.$$

From here, summing over the mesh ω we obtain for $\varepsilon > 0$ and $\alpha > 0$

$$(17) \qquad \|\psi_0\|_\omega \le C \, \|a_0\|_{L_2(\Omega)} \|u\|_{L_\infty(\Omega)} \le C \, \|a_0\|_{L_{2+\varepsilon}(\Omega)} \|u\|_{W_2^{1+\alpha}(\Omega)}.$$

Transforming $u(x_1, x_2) - u(\xi_1, \xi_2)$ in (16) to integral form

$$u(x_1, x_2) - u(\xi_1, \xi_2) = \int_{\xi_1}^{x_1} D_1 u(\tau_1, \xi_2) \, d\tau_1 + \int_{\xi_2}^{x_2} D_2 u(x_1, \tau_2) \, d\tau_2$$

we obtain $\psi_0 = \psi_{01} + \psi_{02}$ where:

$$\psi_{01} = \tfrac{1}{h^2} \iint_e \int_{\xi_1}^{x_1} \Phi(\xi_1, \xi_2) a_0(\xi_1, \xi_2) D_1 u(\tau_1, \xi_2) \, d\tau_1 d\xi_1 d\xi_2 \quad \text{and}$$
$$\psi_{02} = \tfrac{1}{h^2} \iint_e \int_{\xi_2}^{x_2} \Phi(\xi_1, \xi_2) a_0(\xi_1, \xi_2) D_2 u(x_1, \tau_2) \, d\tau_2 d\xi_1 d\xi_2$$

Finally, applying Hölder's inequality, traces theorems [1], and using embeddings $W_2^2 \subset W^1_{2p/(p-2)}$ and $W_2^2 \subset W^{1+(p-2)/2p}_{2p/(p-2)}$ we obtain

$$(18) \qquad \|\psi_0\|_\omega \le Ch\|a_0\|_{L_{2+\varepsilon}(\Omega)}\|u\|_{W_2^2(\Omega)}, \quad \varepsilon > p-2,\ p>2.$$

The mapping $(a_0, u) \to \psi_0$ is bilinear. From (17) and (18) follows that is a bounded bilinear operator from $L_{2+\varepsilon}(\Omega) \times W_2^{1+\alpha}(\Omega)$ to $L_2(\omega)$ and from $L_{2+\varepsilon}(\Omega) \times W_2^2(\Omega)$ to $L_2(\omega)$. Applying Lemma 1, from (17) and (18) it follows that ψ_0 is a bounded bilinear operator from $[L_{2+\varepsilon}(\Omega), L_{2+\varepsilon}(\Omega)]_\theta \times \left[W_2^{1+\alpha}(\Omega), W_2^2(\Omega)\right]_\theta$ to $L_2(\omega)$, with the norm $M \le C \cdot h^\theta$. According to Lemma 2, (6) and setting $\theta = s-1$, we obtain

$$(19) \qquad \|\psi_0\|_\omega \le Ch^{s-1}\|a_0\|_{L_{2+\varepsilon}(\Omega)}\|u\|_{W_2^{s+\alpha(2-s)}(\Omega)}, \quad 1 \le s \le 2.$$

Analogous estimates like (19) hold true for other terms and so we have

$$(20) \quad \|\psi_i\|_\omega \le Ch^{s-1}\|a_i\|_{W_p^{s-1+\alpha(2-s)+\delta}(\Omega)}\|u\|_{W_2^{s+\alpha(2-s)}(\Omega)}, \quad 1 \le s \le 2$$

$$(21) \qquad \|\psi\|_\omega \le Ch^{s-1}\|a\|_{W_2^{s-2+\varepsilon(3-s)}(\Omega)}\|u\|_{W_2^s(\Omega)}, \quad 2 \le s \le 3.$$

Combining (14), (15) and (17)–(19) we have just proved:

Theorem 1. *The FDS (8) converges in the norm of the space $W_2^1(\omega)$ and following estimate, consistent with the smoothness of data, holds true*

$$\|u-v\|_{W_2^1(\omega)} \le Ch^{s-1}(\max_{ij}\|a_{ij}\|_{W_2^{s-1+\varepsilon(3-s)}(\Omega)} + \|a\|_{W_2^{s-2+\varepsilon(3-s)}(\Omega)})\|u\|_{W_2^s(\Omega)},$$

for $2 \le s \le 3$, and

$$\|u-v\|_{W_2^1(\omega)} \le Ch^{s-1}(\max_{ij}\|a_{ij}\|_{W_p^{s-1+\varepsilon}(\Omega)} + \max_i\|a_i\|_{W_p^{s-1+\alpha(2-s)+\delta}(\Omega)} + \|a_0\|_{L_{2+\varepsilon}(\Omega)})\|u\|_{W_2^{s+\alpha(2-s)}(\Omega)}, \quad \textit{for } 1 \le s \le 2.$$

Using inequality (see [5]) $|z|_{W_2^2(\omega)} \le \sqrt{6}/h \cdot |z|_{W_2^1(\omega)}$ and embeddings $W_2^2(\Omega) \subset W_2^3(\Omega)$ and $W_2^3(\Omega) \subset W_2^4(\Omega)$ we can easily prove

Theorem 2. *The FDS (8) converges in the norm of the space $W_2^2(\omega)$ and the following estimate, consistent with the smoothness of data, holds true*

$$\|u-v\|_{W_2^2(\omega)} \le Ch^{s-2}(\max_{ij}\|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)})\|u\|_{W_2^s(\Omega)},$$

for $3 \leq s \leq 4$, *and*

$$\|u-v\|_{W_2^2(\omega)} \leq Ch^{s-2}(\max_{ij}\|a_{ij}\|_{W_2^{s-1+\epsilon(3-s)}(\Omega)} + \|a\|_{W_2^{s-2+\epsilon(3-s)}(\Omega)})\|u\|_{W_2^s(\Omega)},$$

for $2 \leq s \leq 3$.

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