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ON THE PATH NUMBER OF A DIGRAPH

Dragan Mašulović
Institute of Mathematics, Faculty of Science, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
e-mail: masul@unsim.im.ns.ac.yu

Abstract

The vertex path number of a digraph is the minimum number of vertex disjoint paths in the digraph which partition the set of vertices of the digraph. The edge path number of a digraph is the minimum number of edge disjoint paths in the digraph which partition the set of edges of the digraph. We show that the vertex and edge path numbers of a digraph are closely related and that the vertex path number of a digraph equals the minimum vertex path number of a spanning tree of the digraph.

Notation and Preliminaries

For the elementary notions of graph theory (such as digraph, weakly connected digraph, spanning tree of a digraph, etc.) the reader is referred to [2].

Let $x_0, x_1, \ldots, x_n$ be vertices of a digraph D. If $x_i \neq x_j$ for $i \neq j$ and $(x_{i-1}, x_i) \in E(D)$ for $i = 1 \ldots n$, then the subgraph of D induced by the set of edges $\{(x_{i-1}, x_i) : i = 1 \ldots n\}$ shall be referred to as a path in D. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a set of paths in D. $\mathcal{P}$ is a vertex path cover of D iff $\{V(P_1), \ldots, V(P_k)\}$ is a partition of $V(D)$. $\mathcal{P}$ is an edge path cover of D iff $\{E(P_1), \ldots, E(P_k)\}$ is a partition of $E(D)$.

$$\mathrm{pn}_v(D) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a vertex path cover of } D\}$$

is the vertex path number of D.

$$\mathrm{pn}_e(D) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is an edge path cover of } D\}$$

is the edge path number of D. Vertex (edge) path cover $\mathcal{P}$ of D is minimal iff $\mathrm{pn}_v(D) = |\mathcal{P}|$ ($\mathrm{pn}_e(D) = |\mathcal{P}|$). The notion of edge path number was introduced in [1] in order to obtain some additional information on tournaments.

Given a digraph $D = (V, E)$, put $X = \{(e_1, e_2) \in E^2 : (\exists x, y, z \in V)(e_1 = (x, y) \wedge e_2 = (y, z))\}$. Digraph $L(D) = (E, X)$ shall be referred to as the line digraph of D. Using the notion of line digraph we can establish a connection between the vertex and edge numbers of a digraph as follows:

Lemma 1. *Let D be a digraph. Then* $\mathrm{pn}_e(D) = \mathrm{pn}_v(L(D))$. □

The Result

We shall now characterize the vertex path number of a digraph D via vertex path numbers of its spanning trees. Let $\mathrm{ST}(D)$ be the set of all the spanning trees of D.

Theorem 1. *Let D be a weakly connected digraph. Then*

$$\mathrm{pn}_v(D) = \min\{\mathrm{pn}_v(T) : T \in \mathrm{ST}(D)\}.$$

Proof. For every spanning digraph S of D we obviously have $\mathrm{pn}_v(D) \leq \mathrm{pn}_v(S)$. Therefore, $\mathrm{pn}_v(D) \leq \min\{\mathrm{pn}_v(T) : T \in \mathrm{ST}(D)\}$.

Let us prove that the other inequality is also true. Let $\{P_1, \ldots, P_k\}$ be a minimal vertex path cover of D. By adding $k-1$ edges to $E(P_1) \cup \ldots \cup E(P_k)$ we shall construct a spanning tree T_k of D such that $k \geq \mathrm{pn}_v(T_k)$. The construction goes inductively.

Put $T_1 := P_1$ and $\mathcal{P}_1 := \{P_2, \ldots, P_k\}$. Suppose we have constructed the digraph T_j and the set $\mathcal{P}_j$. Since D is a weakly connected digraph, there are vertices $u \in V(T_j)$ and $v \in \bigcup_{P \in \mathcal{P}_j} V(P)$ such that $(u, v) \in E(D)$ or $(v, u) \in E(D)$. Suppose $(u, v) \in E(D)$ and $v \in P_{s_j} \in \mathcal{P}_j$. Let T_{j+1} be a subgraph of D induced by the set of edges $E(T_j) \cup \{(u, v)\} \cup E(P_{s_j})$ and put $P_{j+1} := \mathcal{P}_j \setminus \{P_{s_j}\}$. One can easily check that

- $T_1, \ldots, T_k$ are trees,
- $V(T_k) = V(D)$, i.e. $T_k \in \mathrm{ST}(D)$, and
- $j \geq \mathrm{pn}_v(T_j)$ for all $j = 1 \ldots k$.

Therefore, T_k is a spanning tree of D such that $k \geq \mathrm{pn}_v(T_k)$. Now, it is quite easy to complete the proof: $\mathrm{pn}_v(D) = k \geq \mathrm{pn}_v(T_k) \geq \min\{\mathrm{pn}_v(T) : T \in \mathrm{ST}(D)\}$. □

Corollary 1. *Let D be a digraph.* $\mathrm{pn}_v(D) = \min_F \mathrm{pn}_v(F)$, *where the minimum is taken over all spanning forests F of D.* □

As a conclusion, let us note that determination of $\mathrm{pn}_v(D)$ is NP-hard ($\mathrm{pn}_v(D) = 1$ iff D has a hamiltonian path, and that is an NP-hard problem). According to lemma 1, determination of $\mathrm{pn}_e(D)$ is also NP-hard.

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References

[1] Alspach B., Mason D. W., Pullman N. J., Path numbers of tournaments, J. Comb. Theory (B) 20(1976), 222–228

[2] Chartrand G., Lesniak L., Graphs & Digraphs, 3rd Ed., Chapman & Hall, London, 1996