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HIGHER ORDER EULER-LIKE METHODS

Miodrag Petković
Faculty of Electronic Engineering, University of Niš
Beogradska 14, 18000 Niš, Yugoslavia

Slobodan Tričković
Faculty of Civil Engineering, University of Niš
Beogradska 14, 18000 Niš, Yugoslavia

Djordje Herceg
Institute of Mathematics, Faculty of Science, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We consider some iterative methods of higher order for simultaneous determination of the polynomial zeros. The proposed methods are based on Euler's third-order method for finding a zero of a given function and involve Weierstrass' correction. We prove that the presented methods have the order of convergence equal to four or more. The accelerated convergence is obatined with negligible number of additional operations, so that these methods possess a very high computational efficiency. The convergence speed is illustrated on a numerical example.

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1. Introduction

Let f be a function such that f' is nonzero in a neighborhood of a zero ζ of f and let f'' be continuous in this neighborhood. The classical Euler method reads

$$(1) \qquad \hat{z} = z - \frac{2f(z)}{f'(z) \pm \sqrt{f'(z)^2 - 2f(z)f''(z)}},$$

where $\hat{z}$ is a new approximation to a zero ζ of f (see Euler [4]). This method can be derived by expanding f in the Taylor series about the origin, dropping third and higher order terms, and solving the obtained quadratic equation.

The aim of this paper is the construction of iterative methods for a simultaneous determination of polynomial zeros based on Euler's method. In Section 3 we start from (1) and derive the basic square-root fourth-order simultaneous method of Euler's type. Using suitable corrections, we accelerate the convergence order of this method to five and six with a negligible number of additional operations so that the accelerated methods possess very high computational efficiency. For this reason, they can be of importance in practical application, especially when they are implemented on parallel computers. In view of their convergence behaviour they can compete with the most efficient simultaneous methods for finding polynomial zeros.

The convergence analysis of the proposed methods is given in Section 4. To demonstrate very fast convergence of the presented methods, we give a numerical example in Section 5.

2. Some preliminaries

Let P be a monic polynomial of the degree n with simple zeros $\zeta_1, \ldots, \zeta_n$, and let $z_1, \ldots, z_n$ be n pairwise distinct approximations to these zeros. We introduce the so-called *Weierstrass' correction*

$$(2) \qquad W_i = W_i(z_1, \ldots, z_n) = \frac{P(z_i)}{\prod_{j \neq i} (z_i - z_j)},$$

which appears in the second order Weierstrass' method [7] (also known in the literature as Durand-Dochev-Kerner method, see [2], [3], [5])

$$(3) \qquad z_i^{(m+1)} = z_i^{(m)} - W_i(z_i^{(m)}) \quad (i \in I_n := \{1, \ldots, n\};\ m = 0, 1, \ldots)$$

for the simultaneous determination of all simple zeros of a polynomial P.

By (2) and Lagrange's interpolation formula applied to the polynomial

$$P(z) - \prod_{j=1}^{n}(z - z_j)$$

of the degree $n-1$, we obtain for all $z \in C$,

$$P(z) = \prod_{j=1}^{n}(z - z_j) + \sum_{k=1}^{n} W_k \prod_{\substack{j=1 \\ j\neq k}}^{n}(z - z_j). \tag{4}$$

Let us define

$$h_i(z) := W_i(z_1, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_n) = \frac{P(z)}{\prod_{j\neq i}(z - z_j)}.$$

Then from (4) we obtain

$$h_i(z) = W_i + (z - z_i)\Big(1 + \sum_{j\neq i} \frac{W_j}{z - z_j}\Big). \tag{5}$$

For any $z \in C \setminus \{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}$ we find

$$h_i'(z) = 1 + \sum_{j\neq i} W_j \frac{z_i - z_j}{(z - z_j)^2}, \qquad h_i''(z) = -2\sum_{j\neq i} W_j \frac{z_i - z_j}{(z - z_j)^3}. \tag{6}$$

In this paper we will use the following notation:

$$S_i = \sum_{j\neq i} \frac{W_j}{\zeta_i - z_j}, \quad G_{k,i} = \sum_{j\neq i} \frac{W_j}{(z_i - z_j)^k} \quad (k = 1, 2, \ldots).$$

From (5) and (6) we calculate at the point $z = z_i$:

$$h_i(z_i) = W_i, \quad h_i'(z_i) = 1 + G_{1,i}, \quad h_i''(z_i) = -2G_{2,i}. \tag{7}$$

Note that any zero ζ_i of P is also a zero of the function $h_i(z)$. Let us observe that for any zero ζ_i of P we obtain from (5) (taking $z := \zeta_i$)

$$\zeta_i = z_i - \frac{W_i}{1 + \sum_{j\neq i} \frac{W_j}{\zeta_i - z_j}} \quad (i \in I_n). \tag{8}$$

Remark 1. Substituting the zero ζ_i on the right-hand side of (8) by its approximation z_i there follows

$$\hat{z}_i = z_i - \frac{W_i}{1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j}} = z_i - \frac{W_i}{1 + G_{1,i}} \quad (i \in I_n). \tag{9}$$

This is the iterative method with cubic convergence proposed by Börsch-Supan [1]. Note that Börsch-Supan's method (9) can be derived by applying Newton's method to the function $h_i(z)$.

3. Euler-like simultaneous methods

Applying Euler's third-order method (1) to the function $h_i(z)$ given by (5), we obtain using (7)

$$\begin{aligned} \hat{z}_i &= z_i - \frac{2h_i(z_i)}{h_i'(z_i) \pm \sqrt{h_i'(z_i)^2 - 2h_i(z_i)h_i''(z_i)}} \\ &= z_i - \frac{2W_i}{1 + G_{1,i} \pm \sqrt{(1 + G_{1,i})^2 + 4W_iG_{2,i}}} \quad (i \in I_n). \end{aligned} \tag{10}$$

The iterative formula (10) is very similar to the following fixed point relation (with ζ_i instead of z_i)

$$\zeta_i = z_i - \frac{2W_i}{1 + G_{1,i} \pm \sqrt{(1 + G_{1,i})^2 + 4W_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\zeta_i - z_j)}}} \tag{11}$$

for all $i \in I_n$. This relation can be easily obtained by rearranging the fixed point relation (8) in the form

$$\frac{W_i}{(\zeta_i - z_i)^2} + \frac{1 + G_{1,i}}{\zeta_i - z_i} - \frac{1}{\zeta_i - z_i}\left(G_{1,i} - \sum_{j \neq i} \frac{W_j}{\zeta_i - z_j}\right) = 0,$$

wherefrom we obtain the quadratic equation in $1/(\zeta_i - z_i)$:

$$\frac{W_i}{(\zeta_i - z_i)^2} + \frac{1 + G_{1,i}}{\zeta_i - z_i} - \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\zeta_i - z_j)} = 0.$$

Solving this equation we get (11).

Starting from the fixed point relation (11) and some approximation c_i to the zero ζ_i under the square root, we obtain the iterative formula

$$\hat{z}_i = z_i - \frac{2W_i}{1 + G_{1,i} \pm \sqrt{(1 + G_{1,i})^2 + 4W_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i - z_j)}}} \tag{12}$$

for all $i \in I_n$. For example, for $c_i := z_i$ we immediately obtain the Euler-like formula (10).

To state algorithms with a great computational efficiency, it is reasonable to choose such an approximation c_i in (12) which does not require additional calculations. In this paper we take the following approximations:

(13)

(i) $c_i^{(1)} := z_i$ (simple approximation);

(ii) $c_i^{(2)} := z_i - W_i$ (Weierstrass' approximation (3));

(iii) $c_i^{(3)} := z_i - W_i/(1 + G_{1,i})$ (Börsch-Supan's approximation (9)).

Since Weierstrass' method (3) has a quadratic convergence and Börsch-Supan's method (9) converges cubically, we have

$$c_i^{(r)} - \zeta_i = O_M((z_i - \zeta_i)^r) = O_M(\varepsilon_i^r) \quad (r = 1, 2, 3), \tag{14}$$

where the symbol O_M denotes that two complex numbers are of the same order magnitude.

Remark 2. According to the discussion presented in [6], the sign "+" should be taken in (10) and (12).

4. Convergence order

The order of convergence of the iterative method (12) with the approximations (13) (i)–(iii) is considered in the following theorem:

Theorem 1. *Let the approximations $z_1, \ldots, z_n$ be sufficiently close to the zeros $\zeta_1, \ldots, \zeta_n$ of P and let $c_i^{(r)}$ be one of the approximations given by* (13). *Then the square-root method* (12) *has the order of convergence $r+3$ ($r = 1, 2, 3$).*

Proof. For simplicity, we introduce the abbreviations

$$X_i = 1 + G_{1,i}, \quad y_i = 4W_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\zeta_i - z_j)},$$

$$Y_i = 4W_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i^{(r)} - z_j)}.$$

From (11) and (12) (taking the sign "+", see Remark 2) there follows

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \frac{2W_i}{X_i + \sqrt{X_i^2 + y_i}} - \frac{2W_i}{X_i + \sqrt{X_i^2 + Y_i}}.$$

After some transformations we obtain

$$\hat{\varepsilon}_i = \frac{2W_i\left(\sqrt{X_i^2 + Y_i} - \sqrt{X_i^2 + y_i}\right)}{\left(X_i + \sqrt{X_i^2 + Y_i}\right)\left(X_i + \sqrt{X_i + y_i}\right)} = \frac{2W_i(Y_i - y_i)}{A_i},$$

that is

$$\hat{\varepsilon}_i = 8W_i^2 \cdot \frac{\displaystyle\sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i^{(r)} - z_j)} - \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\zeta_i - z_j)}}{A_i}, \tag{15}$$

where the denominator A_i is given by

$$A_i = \left(X_i + \sqrt{X_i^2 + Y_i}\right)\left(X_i + \sqrt{X_i^2 + y_i}\right)\left(\sqrt{X_i^2 + Y_i} + \sqrt{X_i^2 + y_i}\right).$$

Let us consider only the numerator in (15) because the denominator A_i is bounded and tends to 8 when $\epsilon \to 0$. Using (14) we obtain

$$\sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i^{(r)} - z_j)} - \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\zeta_i - z_j)}$$
$$= (\zeta_i - c_i^{(r)}) \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i^{(r)} - z_j)(\zeta_i - z_j)}$$
$$= O_M(\varepsilon_i^r) \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(c_i^{(r)} - z_j)(\zeta_i - z_j)}.$$

Since $(z_i - z_j)(c_i^{(r)} - z_j)(\zeta_i - z_j) \to (\zeta_i - \zeta_j)^3$ as $\epsilon \to 0$ and $W_i = O_M(\varepsilon_i)$, from (15) there follows $\hat{\varepsilon}_i = O_M(\varepsilon_i^{r+2}\epsilon) = O_M(\varepsilon_i^{r+3})$, which proves the theorem. □

5. Numerical example

To demonstrate the convergence rate of the presented Euler-like iterative methods, we considered the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with the zeros -3, ± 1, $\pm 2i$, $\pm 2i \pm i$. The following complex numbers were taken as initial approximations to these zeros:

$$\begin{aligned} z_1^{(0)} &= -3.3 + 0.2i, & z_2^{(0)} &= -1.2 - 0.3i, & z_3^{(0)} &= 0.2 + 1.7i, \\ z_4^{(0)} &= -1.8 + 1.3i, & z_5^{(0)} &= -1.8 - 0.7i, & z_6^{(0)} &= 2.3 + 1.2i, \\ z_7^{(0)} &= 1.8 - 0.7i, & z_8^{(0)} &= 1.2 + 0.3i, & z_9^{(0)} &= 0.2 - 2.3i. \end{aligned}$$

We tested the iterative methods (12) for the approximations (13) ($r = 1, 2, 3$), executing two iterations. For comparison, we have also applied Weierstrass' method (3). The maximal errors

$$\max_i |\epsilon_i^{(m)}| = \max_i |z_i^{(m)} - \zeta_i| \quad (m = 1, 2)$$

are displayed in Table 1. The denotation $A(-q)$ means $A \times 10^{-q}$. From this table we observe very fast convergence of the accelerated square-root methods $(12)_{r=2}$ of the order 5 and $(12)_{r=3}$ of the order 6. On the other hand, Weierstrass' method (3) converges rather slowly in the first two iterations. After six iterations this method gave $\max_i |\epsilon_i^{(6)}| = 6.95(-9)$.

Method	(3)	(12), r=1	(12), r=2	(12), r=3
$\max \lvert\epsilon_i^{(1)}\rvert$	5.37(-1)	4.16(-2)	9.91(-3)	5.42(-3)
$\max \lvert\epsilon_i^{(2)}\rvert$	3.59(-1)	9.72(-7)	2.28(-11)	4.44(-16)

Table 1 The maximal errors for two iterations. $A(-q)$ means $A \times 10^{-q}$.

Remark 3. As shown in [6], starting from the fixed point relation (11) we can construct an iterative method of Euler's type for the simultaneous inclusion of all simple zeros of a polynomial. This method produces interval approximations that contain the exact zeros, providing automatically not only the error bounds but also taking into account the rounding errors without altering the basic structure of the interval formula.

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