NOVI SAD J. MATH.
VOL. 28, NO. 3, 1998, 145-158

GENERALIZED FUNCTIONS AND THEIR APPLICATIONS

Bogoljub Stanković
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In the first part of the paper we focus on the different reasons for introducing generalized functions. The second part is devoted to different classes of generalized functions. We follow three directions: Generalized functions as continuous functionals defined on some spaces of smooth functions or on a space of holomorphic functions; cohomological definition; generalized functions as elements of a field or a ring which extend the ring of continuous functions or the ring of smooth functions. We also analyze some properties of the mentioned spaces of generalized functions and the possibilities of solving partial differential equations in them.

AMS Mathematics Subject Classification (1991):
Key words and phrases:

1. Introduction

The introduction and developement of mathematical structures led to an important economy of mathematical thinking, but it also led to the creation of more general mathematical tools which were able to meet increasing demands coming from other parts of the science and from the growing number of users.

The second half of the 20th century was a period of a great advancement of science in general, and one expected that mathematics will surmount the limitations of traditional analysis. However, the latter could not offer solutions for the various mathematical models, or give justifications for some elements (for example δ distribution), or accept some operations with abstract elements (for example with derivatives or some divergent integrals) introduced by physicists or engineers. This gave rise to the introduction of generalized functions, whose theory and applications are presently being very developed.

A model of a phenomenon or proces consists of operations which are very restrictive; mostly they are given by partial derivatives or integrals. Therefore,in classical analysis solutionsof such models had to have very special properties. The natural solutions related to these phenomena or processes failed sometimes to satisfy the model, for being without such properties.

The first ideas to overcome this situation were to introduce approximate and weak solutions to partial differential equations and finite part of divergent integrals. But in all these cases one remained in the narrow frame of the classical analysis.

A real progress has been made by introducing generalized functions. Let us analyze the objective and the concept of the introduction of generalized functions.

First, one has to choose a class of numerical functions which have to be generalized. Such a class is $\mathbf{L}_{loc}$ (locally integrable functions), because it is very reach; we do not know how to construct a bounded function which is not Lebesgue integrable in a compact set in $\mathbf{R}^n$.

Second, find a set $\mathbf{M}$ with an algebraic structure defined by a linear and at least a multiplicative operation, which contains a subset $\mathbf{N}$ isomorphic to $\mathbf{L}_{loc}$. The restriction on $\mathbf{L}_{loc}$ of the linear and multiplicative operations should be the addition and the multiplication in $\mathbf{L}_{loc}$. Moreover, the "derivate" in $\mathbf{N}$ should be defined as a continuous operation, the restriction of it giving on $\mathbf{C}^1$ (set of functions with continuous derivatives) the classical derivative.

The set $\mathbf{M}$ has not to be too reach, but it should be large enough to contain abstract elements used by physicists, as $\delta, \delta^2, ...$, to be complete and the mentioned operations be continuous.

In the creation of such a new set $\mathbf{M}$ with its elements and operations three directions were essantial importance:

- Generalized functions as continuous functionals defined on some basic vector spaces of smooth functions, which extend the vector space of locally integrable functions.

- Cohomological definition of generalized functions.

- Generalized functions as a field or a ring which extends the ring of $\mathbf{L}_{loc}$, the ring of continuous functions, or the ring of smooth functions.

The different classes of generalized functions have given step by step a rigorous mathematical sense to some new objects or new operations with them, introduced by physicists. But, an ideal set **M** with all the properties we sought has not been constructed yet. Therefore, we shall list some most important spaces of generalized functions with their properties and imperfections.

2. Generalized functions as linear continuous functionals

We shall start with the class of generalized functions defined as dual spaces of some basic spaces. Clearly, the smaller is the basic space, the more continuous functionals will be in its dual space.

2.1. Distributions

We shall first analyse Schwartz's distributions (see [19]).

Let Ω be any open set in $\mathbf{R}^n$. We denote by $\mathbf{D}(\Omega)$ the vector space of smooth functions φ with suppφ compact set in $\mathbf{R}^n$ belonging to Ω (suppφ is the closure of the set $\{x \in \Omega; \varphi(x) \neq 0\}$. The topology in $\mathbf{D}(\Omega)$ is defined in such a way that $\{\varphi_j\} \subset \mathbf{D}(\Omega)$ converges to zero as $j \to \infty$ if there exists a compact set $K \subset \Omega$ such that supp$\varphi_i \subset K$ for every $\varphi_i \in \{\varphi_j;\ j \in \mathrm{N}\}$ and every partial derivative

$$D^p\varphi_j(x) = D^{p_1+\ldots+p_n}_{x_1,\ldots,x_n}\varphi_j(x) \to 0,\ \ j \to \infty,$$

uniformly in $x \in K$, where $p = (p_1, ..., p_n) \in (\mathrm{N} \cup 0)^n$.

The space of distributions $\mathbf{D}'(\Omega)$ is the strong dual of $\mathbf{D}(\Omega)$; elements of $\mathbf{D}'(\Omega)$ are continuous linear functionals on $\mathbf{D}(\Omega)$. We write for $f \in \mathbf{D}'(\Omega)$: $\varphi \to \langle f, \varphi \rangle$. $\mathbf{D}'(\Omega)$ is not empty; $\mathbf{L}_{loc}(\Omega) \subset \mathbf{D}'(\Omega)$, every $f \in \mathbf{L}_{loc}$ defines a distribution $\bar{f}$

$$\langle \bar{f}, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \varphi \in \mathbf{D}(\Omega).$$

In $\mathbf{D}'(\Omega)$ there are elements which are not defined by functions. Such an element is the δ-distribution: $\varphi \to \varphi(0)$.

By the axiom of choice one can prove the existence of linear non-continuous functionals, but not a single one has been constructed to this day.

If $\Omega = \mathbf{R}^n$, then we write for short $\mathbf{D}'(\mathbf{R}^n) = \mathbf{D}'$.

The support (the singular support) of a distribution $f \in \mathbf{D}'(\Omega)$ is the set of $x \in \Omega$ for which do not exist an open neighbourhood $V(x)$ such that $f(x) = 0 \quad (f(x) = \psi(x);\ \psi(x) \in \mathbf{C}^\infty(\Omega), x \in V(x))$. We write for short $\mathrm{supp} f(\mathrm{sing\ supp} f)$.

If for the basic space we take the whole of $\mathbf{C}^\infty(\Omega) \supset \mathbf{D}(\Omega)$, then the dual space is $\mathbf{E}'(\Omega) \subset \mathbf{D}'(\Omega)$, where $\mathbf{E}'(\Omega)$ is the space of all distributions with compact supports belonging to Ω.

Another subspace of $\mathbf{D}'$ which is often used is the space $\mathbf{S}'$ of tempered distributions. In this case the basic space is

$$\mathbf{S} = \{\varphi \in \mathbf{C}^\infty;\ \lim_{|x| \to \infty} |x^k D^p \varphi(x)| \to 0, \text{ for every } k, p \in \mathbf{N}_0^n = (\mathbf{N} \cup 0)^n\}.$$

The basic properties of $\mathbf{D}'$ are:

1. $\mathbf{L}_{loc} \subset \mathbf{D}'$.

2. If $\psi \in \mathbf{C}^\infty$ and $f \in \mathbf{D}'$, then the product $\psi f : \varphi \to \langle \psi f, \varphi \rangle$ for every $\varphi \in \mathbf{D}$ is defined by $\langle \psi f, \varphi \rangle = \langle f, \psi\varphi \rangle$. We point out that we can multiply a distribution only by a smooth function.

L. Schwartz [18] has proved the impossibility (in a precise sense) of the multiplication of distributions, i.e. the non-existence of a differential algebra $\mathbf{A}$ containing, the algebra $\mathbf{C}(\mathbf{R})$ as a subalgebra, preserving the differentation of functions of class $\mathbf{C}^1(\mathbf{R})$ with Leibnitz's rule and the constant 1 as the neutral element in $\mathbf{A}$.

3. Every distribution has all derivatives and they are continuous; they have been defined by: $\langle D^p f, \varphi \rangle = \langle f, D^p \varphi \rangle$, $p = (p_1, ..., p_n) \in \mathbf{N}_0^n$. If the distribution f is defined by a function which has a first partial derivative $\frac{\partial f}{\partial x_i}$, $1 \le i \le n$, then the derivative $D_{x_i} f$ is defined by the function $\frac{\partial f}{\partial x_i}$.

For any generalized function f we have:

$$D_{x_i}(D_{x_j} f) = D_{x_j}(D_{x_j} f) = D^2_{x_i,x_j} f.$$

If $\psi \in \mathbf{C}^\infty$, the Leibnitz formula holds true for ψf.

Let $\{u_k\} \subset \mathbf{L}_{loc}(\Omega)$. If the series

$$\sum_{k=0}^{\infty} u_k(x) = f(x), \quad x \in \Omega,$$

converges uniformly on every compact set $K \subset \Omega$, then it can be derived term by term and the obtained series converge in $\mathbf{D}'(\Omega)$, as well.

How much we have extended $\mathbf{L}_{loc}$ by $\mathbf{D}'$ it has been precisely stated by the following theorem

Theorem *[19] Any distribution is locally a partial derivative of a continuous function.*

From this theorem it follows that distributions constitute of the smallest vector space in which is permitted to differentiate infinite times all continuous and locally integrable functions.

4. Differential equations in $\mathbf{D}'$.

Theorem *[19, T I, p. 130] For every $(B_1, ..., B_m) \in \mathbf{D}' \times ... \times \mathbf{D}'$ and $A_{j,k} \in \mathbf{C}^\infty$, $j, k = 1, ..., m$, the system*

$$\frac{dT_j}{dx} + \sum_{k=1}^{m} A_{jk}(x) T_k = B_j, \;\; j = 1, ..., m,$$

has infinite solutions. The difference between two of them is a solution of the homogeneous system, and it is a smooth function, the usual solution of the homogeneous system.

If B_i, $i = 1, ..., m$, are continuous functionous, then the system has only classical solutions.

A differential equation which has no classical solutions is the following

$$x^n u^{(m)}(x) = 0, \;\; n > m \ge 1.$$

But it has a solution in $\mathbf{D}'$ ([22], p. 127)

$$u(x)=\sum_{k=0}^{m-1} c_k H(x) x^{m-1-k}+\sum_{k=0}^{n-1} c_k \delta^{(k-m)}(x)+\sum_{k=0}^{m-1} d_k x^k,$$

where c_k and d_k are any constants and H is the Heaviside function.

Consequently, for differential equations the theory of distributions do not bring much novelty. That is the theory of partial differential equations which had an important influence on the developement of distributions and other generalized functions.

5. Partial differential equations in $\mathbf{D}'$ (with constant coefficients).

We denote by $P(D)$ a linear differential operator with constant coefficients

$$P(D)=\sum_{|i|\leq m} a_i D^i, \quad \sum |a_i| \neq 0, \ \ i=(i_1,...,i_n)\in (N\cup 0)^n,$$

where $|i|=i_1+...+i_n$.

Definition *[3, T II, p. 50, 55] An open set Ω is called $P-$convex for the support (for the singular support) if for every compact set $K\subset\Omega$ there exists a compact set $K'\subset\Omega$ such that for every $\varphi\in C^\infty(\Omega)$ ($\psi\in E'(\Omega)$) and* $\operatorname{supp}P(-D)\varphi\in K$ *(sing* $\operatorname{supp}P(-D)\psi\in K$), *it follows that* $\operatorname{supp}\varphi\subset K'$ *(sing* $\operatorname{supp}\psi\subset K'$).

Theorem *[3, T II, p. 61] The equation $P(D)u=f$ has a solution $u\in\mathbf{D}'(\Omega)$ for every distribution $f\in\mathbf{D}'(\Omega)$ if and only if Ω is $P-$convex for the support and for the singular support, as well.*

Theorem. *(Malgrange-Ehrenpreis). Every differential operator with constant coefficients $P(D)\neq 0$ has a fundamental solution in $\mathbf{D}'$, that is, there exists a solution of the equation $P(D)E=\delta$.*

Theorem. *[22, p. 196] Let E be a fundamental solution for the operator $P(D)$, and let the convolution $E*f$ exists for $f\in\mathbf{D}'$. Then there exists a solution in $\mathbf{D}'$ to equation $P(D)u=f$, and it is of the form $u=E*f$.*

6. General linear partial differential equation in $\mathbf{D}'$.

For most systems of linear partial differential equations with variable coefficients distributions solutions do not exist. One of the first example, a very simple one, has been given by H. Lewy [10]. The following equation:

$$-\frac{\partial u}{\partial x_1} - i\,\frac{\partial u}{\partial x_2} + 2i(x_1 + ix_2)\frac{\partial u}{\partial x_3} = f(x)$$

has no solution in $\mathbf{D}'(V_0)$ for any $f \in \mathbf{C}^\infty$, but not analytic, and any point $x_0 \in \mathbf{R}^3$, where V_0 is an open neighbourhood of x_0.

The relation between generalized solutions belonging to $\mathbf{D}'$ and classical ones is given by

Theorem. *[22, p. 193] If $f \in C(\Omega)$ and if a solution u in $\mathbf{D}'$ to the equation*

$$\sum_{|i|\leq m} a_i(x)D^i u = f(x) \tag{2}$$

belongs also to $\mathbf{C}^m(\Omega)$, then it is a classical solution of this equation in Ω.

7. The Fourier transform of $f \in \mathbf{S}'$ is defined by $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle$, $\varphi \in \mathbf{S}$.

The Fourier transform is a linear isomorphism $\mathbf{S}'$ onto $\mathbf{S}'$. It is a continuous operation.

We do not know how to define the Fourier transform for every distribution.

2.2. Ultradistributions

The space of ultradistributions is the dual space of the space of a non-quasi-analytic class of smooth functions.

Let M_p, $p = 0, 1, ...$, be a sequence of positive numbers. We impose the following conditions on M_p : $M_0 = M_1 = 1$ and

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p = 1, 2, ...$

$(M.2)$ There are constants A and H such that

$M_p \leq AH^p M_q M_{p-q}$, $0 \leq q \leq p$, $p = 0, 1, ...$

$(M.3)'$ $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p}$.

Definition. *([5]). Let $K \subset \mathbf{R}^n$ be a compact set and let $h > 0$. We denote by $\mathbf{D}_K^{\{M_p\}}$ the space of all $f \in \mathbf{C}^\infty(\mathbf{R}^n)$ with support in K which* *satisfies*

$$\|D^\alpha f\|_{\mathbf{C(K)}} \le C h^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, \dots$$

for some constant $C > 0$. *Then*

$$\mathbf{D}^{(M_p)}(\Omega) = \operatorname*{ind\,lim}_{K \subset\subset \Omega} \operatorname*{proj\,lim}_{h \to 0} \mathbf{D}_K^{\{M_p\},h},$$

$$\mathbf{D}^{\{M_p\}}(\Omega) = \operatorname*{ind\,lim}_{K \subset\subset \Omega} \operatorname*{ind\,lim}_{h \to 0} \mathbf{D}_K^{\{M_p\},h}.$$

The elements of $\mathbf{D}^{(M_p)'}(\Omega)$ will be called the ultradistributions of Beurling type and the elements of $\mathbf{D}^{\{M_p\}}(\Omega)$ ultradistributions of Roumieu type. Let us denote by $*$ either (M_p) or $\{M_p\}$. Similarly, we obtain the space $\mathbf{E}^{*\prime}(\Omega)$ (see [5]).

An operator of the from

$$P(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha, \quad a_\alpha \in \mathbf{C} \tag{2}$$

is called an ultradifferential operator of class (M_p) (of class $\{M_p\}$) if there are constants L and C (for every $L > 0$ there is a constant C) such that

$$|a_\alpha| \le C L^{|\alpha|} / M_{|\alpha|}, \quad |\alpha| = 0, 1, \dots$$

$\mathbf{D}^{*\prime}$ contains $\mathbf{D}'$ but its new properties are not very different. The most interesting new property is that we can introduce ultradifferential operators in $\mathbf{D}^{*\prime}$ as well as operations with them.

Theorem. *([5] Theorem 6.8). Let $P(D)$ be an ultradifferential operator of class $*$, then it maps $\mathbf{D}^{*\prime}$ into $\mathbf{D}^{*\prime}$, it is a linear and continuous mapping and*

$$P(D)f = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha f,$$

where the series converges absolutely in $\mathbf{D}'^{*}$.

3. Cohomological definition of generalized functions

3.1. Hyperfunctions

Hyperfunctions have been defined in quite different ways. We shall use Sato's cohomological definition ([16], [17]). Denote by $\mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$ the n–th derived sheaf of the sheaf of holomorphic functions $\mathbf{O}$, then the sheaf of hyperfunctions $\mathbf{B}$ is, by definition, $\mathbf{B} = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$, regarded as a sheaf on $\mathbf{R}^n$.

A proof that the sheaf $\mathbf{D}'$ of distributions and the sheaf $\mathbf{D}^{(M_p)}$ of ultradistributions of the Gevrey class ($M_p = (p!)^s$, $s > 1$) of the Beurling type are subsheaves of the sheaf $\mathbf{B}$ of hyperfunctions one can find in [6].

In order to discuss the new possibilities of hyperfunctions we shall mention only some results from the theory of differential equations in $\mathbf{B}$.

Theorem *([6]). Let*

$$P(x, \frac{d}{dx}) = \sum_{i=0}^{m} a_i(x) \frac{d^i}{dx^i}, \quad a_i \in \mathbf{A}(\Omega) \tag{3}$$

be an ordinary differential operator and $\mathbf{A}$ *be the space of real analytic functions. For any* $f \in \mathbf{B}(\Omega)$, *there is a solution* $u \in \mathbf{B}(\Omega)$ *to the equation*

$$P(x, \frac{d}{dx}) u(x) = f(x), \quad x \in \Omega. \tag{4}$$

A direct consequence that $\mathbf{B}$ is a flabby sheaf is:

Corollary. *If* $\Omega' \subset \Omega$ *is also an open set in* $\mathbf{R}$, *then any solution* $u \in \mathbf{B}(\Omega')$ *to (4) can be extended onto a solution* $\tilde{u} \in \mathbf{B}(\Omega)$ *on* Ω.

For a linear differential operator (1) with constant coefficients we have.

Theorem *([7]). Every partial differential equation* $P(D)u = f$ *has a solution in* $\mathbf{B}$ *for any* $f \in \mathbf{B}$.

For a partial differential equation defined by linear differential operators with the coefficients belonging to $\mathbf{A}$, P. Shapira [20] gave a contreexample similar the mentioned one by H. Lewy for distributions.

3.2. Fourier and Laplace hyperfunctions

Fourier hyperfunctions have been defined in various ways (see [4], [16],[20]). They can be also defined as linear continuous functionals. We shall use Sato's definition, linking them to the adopted definition of hyperfunctions.

Denote by $\mathbf{D}^n$ the radial compactification of $\mathbf{R}^n$, $\hat{\mathbf{O}}$ will be the sheaf on $\mathbf{D}^n + i\mathbf{R}^n$ defined as follows: For any open set $U \subseteq \mathbf{D}^n + i\mathbf{R}^n$, $\hat{\mathbf{O}}(U)$ consists of those elements of $\mathbf{O}(U \cap \mathbf{C}^n)$ which satisfy $|F(z)| \le C_{V,\epsilon}\exp(\epsilon|Rez|)$ uniformly for any open set $V \subset \mathbf{C}^n, \bar{V} \subset U$ and for every $\epsilon > 0$. Then the sheaf of Fourier hyperfunctions $\mathbf{Q} = \mathbf{H}^n_{\mathbf{D}^n}(\hat{\mathbf{O}})$.

The Fourier transform acts as a topological automorphism of $\mathbf{Q}$. The space $\mathbf{Q}$ extends the space $\mathbf{S}'$ of tempered distributions and $\mathbf{S}' \hookrightarrow Q(\hookrightarrow$ means a continuous imbeding).

Komatsu ([8] and [9]) introduced another subsheaf of hyperfunctions, the sheaf of Laplace hyperfunctions, in order to make foundation of the Heaviside calculus.

4. Algebras of generalized functions

Let us remark that all the spaces we listed are only vector spaces. We shall now define generalized functions with a richer algebraic structure.

4.1. Mikusinski's field of genralized functions

In the set $\mathbf{L}_{loc}[0,\infty)$ the convolution $f * g = \int_0^t f(t-u)f(u)du$ is defined. With operations $+$ and $*, L_{loc}[0,\infty)$ is a commutative ring without divisors of zero. It can be extended to a field $\mathbf{M}$, the field of Mikusiński operators [11] and [12]. We know that there exist elements of $\mathbf{M}$ which are not distributions and distributions which are not in $\mathbf{M}$. But $\mathbf{M}$ and **D'** have a common part. $\mathbf{M}$ has a rich algebraic structure, but very poor topological one. With the field $\mathbf{M}$ we have a rigorous mathematical theory for Heaviside calculus and a generalization of the classical Laplace transform [12]. But for partial differential equation this theory does not give much, especially if we have boundary conditions.

4.2. Rosinger's algebraic view and Colombeau's new generalized functions

Two very closed ideas have been elaborated shedding more light on the nonlinear problems, Rosinger's algebraic view and Colombeau's new generalized functions.

Let

$$T(D) = \sum_{1\le i\le h} c_i(x) \prod_{1\le j\le k_i} D^{p_{i,j}}, \quad c_i \in \mathbf{C}^\infty(\mathbf{R}^n), 1 \le i \le h.$$

Rosinger ([15] Chapter 6) constructed a chain of algebras $\{\mathbf{A}^j; j \in \bar{N}_0\}$ such that

$$\mathbf{D}'(\Omega) \subset \mathbf{A}^\infty \to \ldots \to \mathbf{A}^j \to \ldots \to \mathbf{A}^0$$

(the arrows mean algebra homomorphisms) and $T(D)$ maps $\mathbf{A}^j$ into $\mathbf{A}^k, j - k \ge m$, where $m = \max|p_{i,j}|, 1 \le i \le h, 1 \le j \le k_i$.

Let $T(D)U(x) = f(x), x \in \Omega, f \in \mathbf{C}^\infty(\Omega)$. We can construct a sequence of smooth functions $s \in (\mathbf{C}^\infty(\Omega))^\wedge$ ($\wedge$ is an arbitrary infinite index set) such that $s \in S^j, j \in N$ and $U_j = s + V^j \subset \mathbf{A}^j + S^j/V^j, j \in \mathbf{N}$. We have then in a sense

$$T(D)U_j = f \in \mathbf{A}^k, j, k \in \mathbf{N}, k + m \le j. \tag{5}$$

The sequence of smooth functions s is called a generalized solution to equation (5). If $\wedge = \mathbf{N}$, then s is called the weak solution to equation (5). For the solutions in this sense see [15, Chapter 7].

For the Colombeau new generalized functions we follow the simplified version as it has been done in [2] and [13], and present some results concerning partial differential equations.

The space $\mathbf{E}_M(\Omega)$ consists of families G_ϵ of smooth functions in Ω, $\epsilon \in (0,1)$, such that for every $K \subset\subset \Omega$ and $\alpha \in \mathbf{N}_0^n$ there exists $N > 0$ such that $\sup|D^\alpha G_\epsilon(x)| = O(\epsilon^{-N})$, $x \in K$; $\mathbf{N}(\Omega)$ consists of families $G_\epsilon \in \mathbf{E}_M(\Omega)$ such that for every $K \subset\subset \Omega$, $\alpha \in N_0^n$ and $r \in \mathbf{R}$ $\sup|D^\alpha G_\epsilon(x)| = O(\epsilon^r)$.

The space of Colombeau's generalized functions, $\mathbf{G}(\Omega)$, is defined by $\mathbf{E}_M(\Omega)/\mathbf{N}(\Omega)$.

If in these definitions we have the families of complex numbers instead of smooth functions, then we obtain the space $\mathbf{E}_{M,0}, \mathbf{N}_0$ and $\bar{\mathbf{C}} = \mathbf{E}_{M,0}/\mathbf{N}_0$ respectively. $\bar{\mathbf{C}}$ is called the space of Colombeau complex numbers.

$G \sim 0$ means that G has a representative G_ϵ such that $G_\epsilon = o(1)$ as $\epsilon \to 0$. If G_1, G_2 are in $\mathbf{G}(\Omega)$, then $G_1 \overset{\mathbf{D}}{\sim} G_2$ if $\langle G_1 - G_2, \varphi\rangle \sim 0$ for every $\varphi \in \mathbf{D}$.

$\mathbf{C}^\infty(\Omega)$ is a subalgebra of $\mathbf{G}(\Omega)$. The Schwartz impossibility result implies that $\mathbf{C}(\Omega)$ can not be a subalgebra of $\mathbf{G}(\Omega)$. $\mathbf{G}(\Omega)$ contains $\mathbf{D}'(\Omega)$. The definition of $\mathbf{G}(\Omega)$ can be modified so as to include $\mathbf{D}^{*'}(\Omega)$ too. An open question is: Is it possible to make this modification of definition of $\mathbf{G}(\Omega)$ such that $\mathbf{G}(\Omega)$ includes also hyperfunctions, [2]?

For the nonlinear partial differential equations in $\mathbf{G}(\Omega)$ one can consult for example [2]. To illustrate we give

Theorem *[14] Let*

$$P_\epsilon(x, D) = a_{m,\epsilon}(x) D_1^m + \sum_{k=0}^{n-1} \sum_{|\alpha| \leq k} a_{\alpha,\epsilon}(x) D^\alpha D_1^k, \ x \in \Omega; \tag{6}$$

we suppose that there exist $C > 0$ and $\eta > 0$ such that $|a_{m,\epsilon}(x)| \geq C\epsilon^N$, $x \in \Omega$, $\epsilon \in (0, \eta)$. Then there exists a solution $G \in G(\Omega)$ to

$$P(x, D)G \overset{H^m(\Omega)}{\sim} H \quad in\ \Omega, \ \ G|_{\varphi\Omega} = 0, \ H \in G(\Omega),$$

where $P_\epsilon(x, D)$ is a representative of $P(x, D)$.

References

[1] Chung, J., Chung, S.Y., Kim, D., A Characterization of Fourier Hyperfunctions, Publ. RIMS, Kyoto Univ. 30, (1994), 203-208.

[2] Colombeau, J.F., Elementary Introduction to New Generalized Functions, North Holand, Amsterdam, 1985.

[3] Hörmander, L., The Analysis of Linear Partial Differential Operators, I, II, Springer Verlag, Berlin, 1983 (Moskva "Mir", 1986).

[4] Kaneko, A., Introduction to Hyperfunctions, Kluwer Academic Publishers, Dordrecht, 1988.

[5] Komatsu, H., Ultradistributions, I, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 20 (1973), 23-105.

[6] Komatsu, H., An introduction to the theory of hyperfunctions, Lecture Notes in Mathematics 287, Springer Verlag, Berlin, 1973.

[7] Komatsu, H., Hyperfunctions and linear partial differential equations, Conference on Generalized Functions, Katowice 1966.

[8] Komatsu, H., Laplace transform of hyperfunctions - A new foundation of the Heaviside calculus, J. Fac. Sci. Univ. Takyo, Sec. IA 34 (1987), 805-820.

[9] Komatsu, H., Laplace transforms, another foundation of the Heaviside operational calculus, Generalized Functions, Convergence Structures and their Applications, Plenum Press, New York, 1988.

[10] Lewy, H., An Example of a Smooth Linear Partial Differential Equation without Solution, Ann. Math., vol. 66, N^0 2 (1957), 155-158.

[11] Mikusiński, J., Sur les fondements du calcul opératoire, Studia Math. 11(195), 41-70.

[12] Mikusiński, J., Operational Calculus, I, II, Pergamon Press 1983.

[13] Pilipović, S., Colombeau's generalized functions and pseudo-differential operators, Lectures in Mathematical Sciences, The university of Tokyo, 1994.

[14] Pilipović, S., Linear equations with singular coefficients in the frame of Colombeau's generalized functions, Pros. II Math. conf. Priština, 1997.

[15] Rosinger, E.E., Non-Linear Partial Differential Equations, An Algebraic View of Generalized Solutions, North-Holland, Amsterdam, 1990.

[16] Sato, M., Theory of Hyperfunctions, Sugaku, 10(1958), 1-27 (In Japanese).

[17] Sato, M., Theory of hyperfunctions, J. Fac. Sci. Univ. Tokyo, Sec. I, 8 (1959), 139-193.

[18] Schwartz, L., Sur impossibilité de la multiplication des distributions, C.R. Ac. Sci. Paris, t. 239 (1954), 847-848.

[19] Schwartz, L., Théorie des distributions, Hermann, Paris I, II, 1951.

[20] Shapira, P., Une équation aux dérivées partielles sans solutions dans l'espace des hyperfunctions, C.R. Ac. Sci. Paris, t. 265 (1967), 665-667.

[21] Stanković, B., Huperfunctions, Zbornik radova 7 (15), Matematički Institut SANU, Beograd (1997), 71-110.

[22] Vladimirov, V.S., Equations of Mathematical Physics, Moskva "Nauka", 1988 (InRussian)

[23] Zharinov, V.V., Laplace transform of Fourier hyperfunctions and other close classes of analytic functionals, Theor. Math. Phys. 33: 3 (1977) 291-309 (In Rusian).